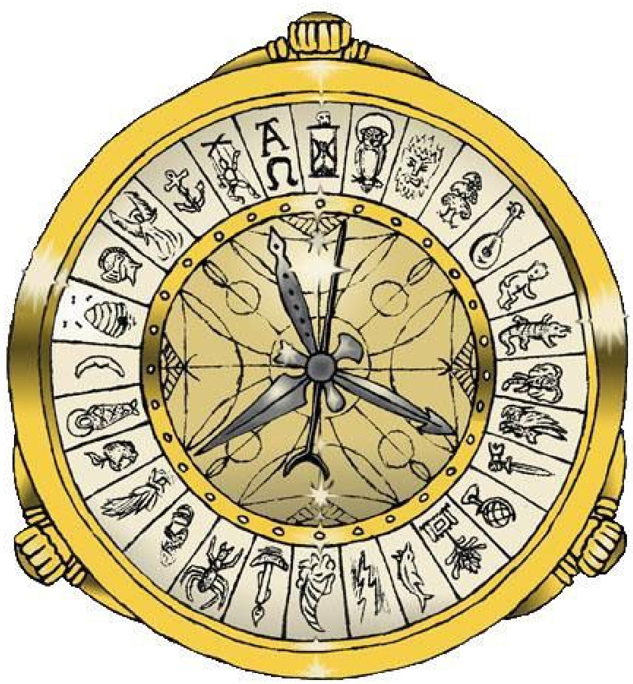
JOY MITRA

CLASSICAL MECHANICS



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Contents

disclaimer

These notes have been used to teach Classical Mechanics to 3rd year BS-MS, 1st year $\int$PhD and MSc and few PhD students at the School of Physics, IISER TVM. They combine material from various texts Marion and Thornton, Goldstein, Hand and Finch and Raychaudhuri and other material available online. But these notes are my take on Classical Mechanics and thus different from each of these sources individually. I typed these notes since the ink of my hand written notes were fading over the years - self preservation, document my own understanding and most importantly the Covid 19 pandemic threw my teaching plan into disarray and I had no clue how to communicate with the students purely through recorded lectures - where I was staring at a camera, without an audience.

The notes are likely riddled with typographical errors - so be forewarned! They are NOT a substitute for standard textbooks and should be read in conjunction with them.

books

1. H. Goldstein, C. Poole and J. Safko, Classical Mechanics, 3rd Ed. Addison Wesley, 2005.
2. L. D. Landau and E. M. Lifshitz, Mechanics, Vol. 1 of course of Theoretical Physics, Pergamon Press, 2000.
3. S. G. Rajeev, Advanced Mechanics, Oxford University Press, 2013
4. Marion and Thornton, Classical Dynamics of Particles and Systems, Cenage Learning
5. D. Wells, Lagrangian Dynamics, Schaum's Outline Series, McGraw Hill
6. A K Raychaudhuri, Classical Mechanics, Oxford University Press
7. Hand and Finch, Analytical Mechanics, Cambridge University Press
8. Simmons and Krantz, Differential Equations, McGraw Hill.

The beauty of physics lies in the extent to which complex and seemingly unrelated phenomena can be explained and correlated through a high level of abstraction by a set of laws which are amazing in their simplicity.

"With four parameters I can fit an elephant, and with five I can make him wiggle his trunk."

John von Neumann

Introduction

The study of mechanics and mechanical motion is ancient, with a rich history of discoveries, both material and fundamental. Its understandings are distilled from a lot of painstaking hard work - with intrigue, conspiracies, rivalries, murder and executions thrown in for good measure, as with any other human endeavours. The labour of a string of individuals, in the early modern era has culminated in what we call today Classical Mechanics. Copernicus, Tycho Brahe, Kepler, Galileo, Descartes, Fermat, Newton, Lagrange and Hamilton were some who made notable contributions to the development of classical mechanics. The contributions of Copernicus, Tycho Brahe and Kepler primarily derived from observing motion of celestial objects. Galileo and importantly Newton reconciled experimental observations with mathematical analysis leading to the development of the physics of mechanics. Development of the mathematical ideas and tools were advanced by the bold ideas of Descartes and Fermat - especially in unifying geometry and algebra giving rise to analytical geometry, which paved the way for independent yet contemporaneous development of calculus by Leibniz and Newton.

Etymology of **algebra**: from the arabic word *al-jabr* in the title of the 9^{th}C book "*Ilm al-jabr wa l-muqābala*", translated as "The Science of Restoring and Balancing" by the Persian mathematician and astronomer *al Khwarizmi*. His name Latinized as *Algorithmi* gave rise to modern terms algorism and algorithm. His book on arithmetic, *Algorithmo de Numero Indorum* introduced the Indian number system and decimals to Europe.

Rene Descartes lends his name to the *Cartesian* coordinate system, though the concept was developed by Fermat a couple of years before Descartes, in the 17th C.

Today, the physical laws of nature are divided into two realms, the Classical and the Quantum. Notionally, the latter encompasses the former, which anyway is applicable only to describe the macroscopic world. However, the role of classical mechanics in development of quantum mechanics is immense. Not only did the failure of the classical laws in describing physical observations lead to the formulation of quantum mechanics - but development of classical mechanics also laid the ground work for the language, mathematics and the analytical techniques used in quantum mechanics. The material that we will cover as a part of this course are still vital to various areas of modern research, from space travel and robotics to chaos and weather prediction. Description of much of the mechanical and electrical universe around us are based on these formulations and understandings.

*To begin with, Classical Mechanics (CM) deals with physical laws describing the motion of "***point***" particles in space. It is also applicable to extended objects like "***rigid bodies***", represented as a continuum - a continuous collection of point particles. A part that deals with (geometrical) description of motion, with out reference to the forces or causes and their* *origins is known as kinematics e.g. Kepler's laws of planetary motion. Dynamics, on the other hand deals with the forces and causal relationships, e.g. the laws expounded by Isaac Newton.*

The Clockwork Universe

Newton's laws introduced determinism or predictability in the mechanical universe and led to the concept of the "clockwork universe", which was popular during the European enlightenment. It is best described in the words of Samuel Clarke;

"The Notion of the World's being a great Machine, going on without the Interposition of God, as a Clock continues to go without the Assistance of a Clockmaker; is the Notion of Materialism and Fate, and tends, (under pretence of making God a Supra-mundane Intelligence,) to exclude Providence and God's Government in reality out of the World."

Here the universe is visualised as a perfect machine, with cogs, gears and wheels governed by the laws of physics, making every aspect of the machine predictable. Though determinism was a new concept, the kinematic reproducibility in the paths of heavenly bodies had been known since antiquity.

The earliest known example of a machine predicting the workings of the universe is the Antikythera mechanism (figure 1), which is an ancient Greek orrery used to predict astronomical positions and eclipses decades in advance. The instrument has been described as the oldest example of an analogue computer and is believed to have been designed and fabricated by the Greeks between 87 BC - 205 BC. It was discovered near a shipwreck off the Greek island of Antikythera in 1901. More than a hundred years later, a team from Cardiff University, UK used computer aided x-ray tomography and imaging to view the inner fragments of the device and read the inscriptions on the machine. The 3D tomography images suggest that the instrument had 37 bronze gears enabling it to follow the movements of the Moon and the Sun through the zodiac, to predict eclipses and even model the irregular orbit of the Moon. Based on the images a working model was reconstructed as shown in figure 2. See the following links for related documentaries and talks;

1. BBC: The Antitkythera Mechanism. 2. Youtube: Stanford Univ.

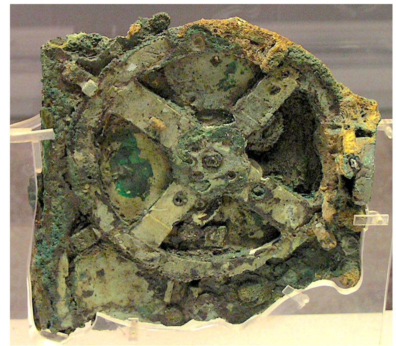


Figure 1: Antikythera mechanism: The largest gear is approximately 13 cm in diameter and has 233 teeth

Figure 2: Antikythera mechanism: A modern reconstruction from 2007

The Reality of Newton's Laws

Refs: Raychaudhuri Chapter 1
Refs: Goldstein, Poole, Safko Chapter 1

In the 17^{th} century, Issac Newton formulated 3 physical laws that essentially laid the foundation of modern mechanics. The laws are deterministic in nature and describe accurately the dynamics of point particles and rigid bodies, under the application of forces. The triumph of Newton lies in the law's relative simplicity and wide applicability. Newton used the same laws to explain the kinematics of earthly objects, e.g. the swinging motion of a pendulum, and to explain the kinematics of heavenly bodies, i.e. Kepler's laws of planetary motion, in combination with Newton's laws of gravity.

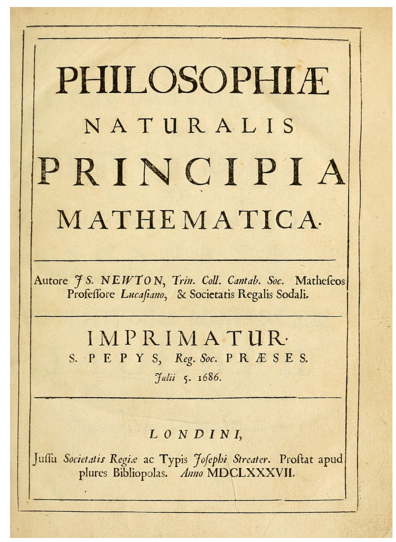

PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.
Autore *JS.* NEWTON, *Trin. Coll. Cantab. Soc.* Mathefeos Profeffore *Lucafiano*, & Societatis Regalis Sodali.
IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.
LONDINI,
Juffu *Societatis Regiæ* ac Typis *Jofephi Streater*. Proftat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Figure 3: Title page of first edition 1687

The Laws as compiled by Isaac Newton in Principia Mathematica.

Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Lex II: Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

Lex III: Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.

Newton's 1st Law

Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

Newton's 2nd Law

The rate of change of momentum of a body is proportional to the force impressed on the body, and happens along the straight line along which that force is impressed.

Newton's 3rd Law

To every action there is always an equal and opposite reaction.

We intuitively think we understand what these laws mean, after all each of us have studied these same laws for at least 8+ years. But, do we fully comprehend the inherent assumptions made in writing the laws, which are required to test their validity and interpret the implications. Let us take a closer look, especially at the underlined phrases in the laws.

1. *What exactly is a 'body'? Surely not all objects qualify, which will satisfy Newton's Laws. Say we take a cube of butter, a block of wood and a*

frog; all sitting at rest (on a very smooth table) and give each the same push. Their 'reactions' will not be the same. Most likely the butter will deform, the block slide forward and the frog croaks, jumps and perhaps do something more to express its displeasure. Idealisation as **point particles** *and* **rigid bodies** *seem to fit the bill. A point particle is one without any internal structure, even a planet is approximated as a point particle for analysing its orbital motion (but not its spinning or rotational motion). An extended body can be thought to be composed of many point particles, leading to a continuum. The notion of rigid body is then imposed by requiring that the distance between any two pairs of point particles is a constant ($|\vec{r}_{ij}| = |\vec{r}_i - \vec{r}_j| = c_{ij}$). Think about the implications of this last statement - it also implies that information can be communicated at infinite speed! Note that the notion of a particle is central to CM, the state of which is fully described in terms of the particle's position and velocity. Specifying the two will let us compute the position and velocity of the particle at all times provided all external influences or forces on the particle are known. So these are the kinematic variables.*

Now consider a time bomb sitting at rest which explodes at a pre-determined time. There was no impressed force to cause the bomb to explode - so how does the first law apply in this case?

2. *What is meant by 'state of rest' or 'moving uniformly in a straight line'? Again you'll appreciate that the state of rest or uniform motion is definitely not absolute, it depends on the state of the observer. Specifically you know that Newton's laws are only valid in certain reference frames called inertial frames of reference. Thus one might say that the 1st law actually defines an inertial frame from which to make the experimental observations that verify Newton's laws. They also define the concept of Galilian Invariance or Newtonian Relativity. Crucially, the 1st law also presupposes an obvious fact that an observer can measure time intervals continuously and absolutely. For without that we cannot measure quantities like velocity or acceleration.*

3. *Now what exactly is a force? The concept of force not only means physical push or pull or contact forces but also non-contact forces, those that act at a distance e.g. gravity, electrostatic attraction or repulsion etc. The gravitational force is regarded as an external force while analysing the motion of a pendulum. But, how does force affect the motion of a body? The 2nd law gives us exactly that, a mathematical formulation of force, namely;*

$$\sum \vec{F}_i = \frac{d\vec{p}}{dt} \tag{1}$$

where the LHS refers to the vector sum of all the forces acting on a body whose linear momentum is defined as $\vec{p} = m\vec{v}$. Assuming that the

The upshot is that given this second order differential equation and specified initial conditions, we can exactly determine the trajectory of the ensuing motion - and then extrapolate it both backwards and forwards in time. This is in principle the be all and end all of Newtonian mechanics.

inertial mass is a constant it gives us,

$$\sum \vec{F}_i = m\vec{a} \tag{2}$$

where $\vec{a}$ is the acceleration of the body. Equation 1 can easily be generalised for a system of particles and also for a system where mass in not a constant.

4. *What is the reality of a force? Consider 2 observers, A standing in an inertial frame and B standing in a box falling freely under gravity. Each will observe that the other is accelerating in the opposite direction, but what do they conclude. A would say that since the box (carrying B) is not at rest or in uniform motion there must be a force acting downward - attributed to the gravitational pull. B will also conclude that A is being acted upon by an upward force - but here the problem starts, that upwards force is not attributable to anything! One might say that both the desperately trying to save Newton's Laws, A does so successfully but B fails! B fails because he violates the* **causal relationship**, *another basic requirement of Newtonian Mechanics. Obviously, all measurements must be done from an inertial frame, else we run into trouble.*

5. *The* **Principle of Superposition** *is evidently implicit in the modern mathematical formulation of the second law; equation 1.*

6. **Conservation of linear momentum.** *The total linear momentum of a system of particles is given by $\vec{P} = \sum \vec{p}_i = \sum m_i d\vec{r}_i/dt$ and $d\vec{P}/dt = \vec{F}_{ext}$, which implies that if $\vec{F}_{ext} = 0$ then $\vec{P}$ remains constant - Conservation of linear momentum in the absence of external forces.*

7. **Conservation of angular momentum** *in the absence of external torque. Consider the total angular momentum of a system of particles, moving under the action of no external forces ($\vec{F}_{ext}$) but finite mutual interactions $\vec{F}_{ij}$.*

$$\begin{aligned}
\vec{L} &= \sum \vec{l}_i = \sum \vec{r}_i \times \vec{p}_i \\
\implies d\vec{L}/dt &= \sum d\vec{r}_i/dt \times \vec{p}_i + \sum \vec{r}_i \times d\vec{p}_i/dt \\
d\vec{L}/dt &= \sum d\vec{r}_i/dt \times \vec{p}_i + \sum \vec{r}_i \times (\sum \vec{F}_i(ext) + \sum \vec{F}_{ij}) \\
d\vec{L}/dt &= \sum d\vec{r}_i/dt \times \vec{p}_i + \sum \vec{r}_i \times \sum \vec{F}_i(ext) + \sum \vec{r}_i \times \sum \vec{F}_{ij}
\end{aligned}$$

The first term in the RHS vanishes due to the cross product and so does the second term in the absence of external forces. Now if the mutual interaction forces, F_{ij} are (i) equal and opposite as given by Newton's third law and (ii) lie along the line joining the particles; then the third term also equals zero. If the external force is non-zero then;

$$d\vec{L}/dt = \sum \vec{r}_i \times \sum \vec{F}_i(ext) = \vec{N}(external)$$

where $\vec{N}$ is the external torque acting on the system and implied that $\vec{L}$ is conserved if $\vec{N}(external) = 0$.

The strong law of action and reaction states that the force that a particle exerts on another is directed along the line joining the two particles, i.e., $(\vec{r}_i - \vec{r}_j) \parallel \vec{F}_{ij}$

8. **Reciprocity**
Let us discuss an apparently non-obvious problem with the 3^{rd} Law. Consider 2 positively charged particles (Q_1 and Q_2) are moving with velocity $\vec{v}_i$ along perpendicular directions, 1 along x and 2 along negative y, as shown in the figure. The electrostatic forces between the charge particles at any instant are repulsive, equal in magnitude and are directed away from each other along the line joining the particles, i.e. they conform to the 3^{rd} law. The reason of course is that the electric force is proportional to $Q1 \times Q2$ and is directed along the vector that points from one charge toward the other. (Similarly, gravitational forces also conform to the 3rd law because the gravitational force is proportional to $m_1 \times m_2$ and is dependent on the vector that points from one mass toward the other)

The magnetic force of interaction is quite different. Q_1 subtends zero magnetic field at the location of Q_2 thus no Lorentz force acts on Q_2. Q_2 however does subtends a finite magnetic field at the location of Q_1, pointing into the page. Thus Q_1 experiences a Lorentz (magnetic) force on it, in the $+y$ direction. The forces acting on the particles 1 and 2 are given by;
$\vec{F}_1 = Q_1\vec{v}_1 \times \vec{B}_2 \neq 0$ *and* $\vec{F}_2 = Q_2\vec{v}_2 \times \vec{B}_1 = 0$
i.e. $\vec{F}_1 \neq \vec{F}_2$. *Clearly the magnetic force does not adhere to the 3rd law. Thus the 3rd law is not fundamental but rather a relationship that applies to electrostatic and gravitational interactions but not to all types of interactions in general.*

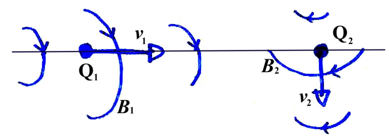


Figure 4: 2 charge particles Q_1 and Q_2. The curved lines denote the magnetic field.

9. *Conservation of linear momentum is commensurate with Newton's third law but conservation of angular momentum has the additional requirement i.e.* $(\vec{r}_i - \vec{r}_j) \parallel \vec{F}_{ij}$. *In the case of the 2 charged particles neither are the interaction forces equal and opposite to each other, nor is the total angular momentum of the 2 particle system a conserved quantity - even in the absence of external forces. We interpret the electromagnetic forces between the particles as internal to the system. The problem is salvaged by redefining momentum by sacrificing its definition as a purely mechanical parameter defined as* $\vec{p} = m\vec{v}$, *but expanding its scope and assigning momentum to the associated electromagnetic field.*

Features of Newton's Laws;

- *Assumes homogeneity of space and time*
- *They are deterministic in nature*
- *Motion is continuous but relative*
- *Time is continuous and absolute*
- *Defines inertial reference frames: frames wherein Newton's Laws are valid*

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."
Pierre Simon Laplace (1749–1827) on the nature of Newton's Laws

- *Concept of Causality and Principle of Superposition*

The above features puts in certain obvious limitations on formulation and application of the laws.

Laws of Physics

Laws of physics are relationships between measurable quantities, enunciating their interdependence. For example, Coulomb's law gives the dependence of the electric force on a point charge on another point charge and the distance between them;

$$\vec{F} = \frac{1}{4\pi\epsilon_o}\frac{Q_1 \times Q_2}{r^2}\hat{r} \tag{3}$$

Though the exact form of any such equation would require experimental validity much can be deduced between the physical dependencies of independent dimensional parameters, indeed even arrive at functional relationships i.e. functional form of a dependent variable on independent variables. Consider the following problem;

In an atomic explosion, a large amount of energy E is released within a small region, and a strong spherical shock wave develops at the point of detonation. Estimate E knowing that the wavefront reaches a distance r at time t.
It is reasonable to assume that the atmospheric pressure is negligible compared to that of the shock wave. Thus we can write, $r = f(E, \rho, t)$, where ρ is the density of air at equilibrium. Using dimensional analysis show that the functional relationship will be of the form;

$$r \propto \frac{E^{1/5} \times t^{2/5}}{\rho^{1/5}} \tag{4}$$

The proportionality constant will be $\mathcal{O}(1)$.

Inertial and Gravitational Mass

Newton also deduced the gravitational force law given by;

$$\vec{F}_{ij} = -Gm_i m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3} \tag{5}$$

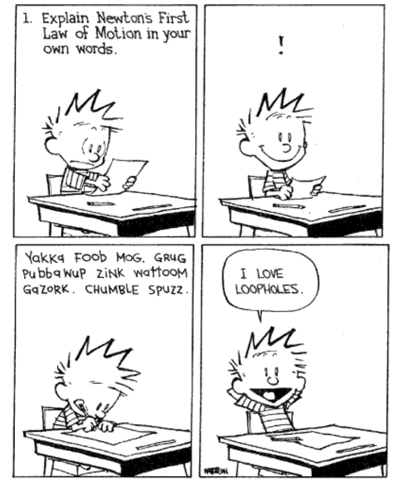
where G = (6.6726 $\pm$ 0.0008) $\times 10^{-11}$ Nm/kg, (first measured by Henry Cavendish in 1798). The functional form guarantees that $F_{ij} = F_{ji}$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, $\vec{F} = m\vec{g}$ where

$\vec{g}$ points towards the earth's center and $g = GM_e/R_e^2 \simeq 9.8m/s^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\vec{a}$=-$\vec{g}$, i.e. objects accelerate as they fall to earth. However, it is not a priori clear why the inertial mass which enters into the definition of momentum should be the same as the gravitational mass which enters into the gravitational force law. If the gravitational mass took a different value, m^* then Newton's Second Law would read;

$$\vec{a} = -\frac{m^*}{m}\vec{g} \tag{6}$$

and unless the ratio m'/m were the same number for all objects, then bodies would fall with different accelerations. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, i.e. $m' = m$.

Homework



1. *Prove that if the forces acting on a particle are conservative i.e. derivable from a scalar potential, then the total energy i.e. kinetic + potential, of the particle is conserved.*

2. *For a system of N particles of masses m_i and position r_i show that;*

$$M\frac{d^2\vec{R}}{dt^2} = \sum \vec{F}_i \tag{7}$$

where $\vec{R}$ is the centre of mass and M total mass of the system of particles.

3. *Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:*

$$\frac{dT}{dt} = \vec{F}.\vec{v} \tag{8}$$

while if the mass varies with time the corresponding equation is;

$$\frac{d(mT)}{dt} = \vec{F}.\vec{p} \tag{9}$$

4. *Consider a vertical disk rolling without slipping on the xy plane. Show that the equations of constraint are given by;*

$$dx - a\sin\theta d\phi = 0 \tag{10}$$

$$dy + a\cos\theta d\phi = 0 \tag{11}$$

5. *The escape velocity of a particle on Earth is the minimum velocity required at Earth's surface in order that the particle can escape from Earth's gravitational field. Neglect any resistance due to the atmosphere and existence of the Moon, From the conservation theorem for potential plus kinetic energy show that the escape velocity for Earth is 11.2 km/s.*

6. *Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is burnt and expelled. Show that the EOM for a rocket projected vertically upward in a uniform gravitational field neglecting atmospheric friction, is;*

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg \tag{12}$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of in, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass loss per second equal to l/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must, be almost 300!

Constraints and Generalized Coordinates

In general, 3 independent space coordinates or degrees of freedom (DOF) are required to uniquely specify the position of a point particle in the 3D space. And for a N particle system we require 3N coordinates. Spatial coordinates in the Cartesian coordinate system have the dimension of unit length, but that is not true for other Euclidean coordinate systems e.g. spherical polar or the cylindrical coordinate systems. Often to exploit the inherent symmetry of a problem we find it convenient to use different coordinate systems, but the biggest casualty of choosing these non-Cartesian systems is sacrificing the simple mathematical form of Newton's 2nd Law;

$$\vec{F} = m\ddot{\vec{r}} \implies m\ddot{x} = F_x; m\ddot{y} = F_y; m\ddot{z} = F_z$$

in the Cartesian system but in the spherical polar system,

$$m\ddot{r} = F_r; m\ddot{\theta} \neq F_\theta; m\ddot{\phi} \neq F_\phi$$

In principle, starting with the 2nd law and the knowledge of all forces acting on a system and proper initial conditions allow us to solve the differential equations to obtain the trajectory of the system. Here the force term F incorporates all forces acting on the system both "external" and "internal" forces. While the external forces may be known upfront the "internal" forces are not known apriori, and often have to be solved for as a part of the solving the problem itself. Such is the case with constraint forces.

Restrictions on the general motion of a system of particles are known as constraints. Ensuring that motion is compliant with the constraints is guaranteed by the corresponding forces of constraint. These forces are often unknown to us a priori, thereby rendering the problem unsolvable upfront. Indeed at times these forces are determined as a part of the solution. We encounter many constraints even in our everyday life. Say your group of friends want to travel from Trivandrum to Kolkata and you are tasked with finding a route. The easiest solution would be to follow a straight line connecting the 2 cities. However, that would require a tunnel to be dug through the earth, which being impractical you will be constrained to restrict *your path to the surface of the earth. But any path along the surface will also not be practical and your group will be restricted to move along a path through the existing highways or rail network connecting the two cities. Here, we encounter a connectivity problem subject to multiple constraints which restricts the number of options and can deliver solution(s) subject to the constraints. Another example would be the cabin baggage allowance of airlines. Some airlines restrict carry-on cabin bags via their dimensions e.g. the sum of its dimensions (length + breadth + height) to a fixed value, say 75 cm. From among the range of bags available in the market passengers are constrained to choose one among a subset of bags that conforms to the above stipulated rule. We will revisit these problems in the next chapter.*

Similarly, in mechanics, the presence of constraints restrict the motion of a particle or system of particles to a subspace or region of the overall space. Let us consider a couple of examples to understand constraints better;

I. *Consider the case of a simple pendulum of length l, that moves in a vertical plane under the action of gravity. The motion of the bob, a point particle that is, (1) constrained to move in the vertical plane would indicate that we need only 2 coordinates (x, y) or (r, θ) to describe its position, additionally (2) the presence of the rigid, massless string constraints the bob to move such that $x^2 + y^2 = l^2$ constant or $r = l$ constant. Here, the presence of the 2 constraints shows that out of the 3 DOF only one is independent and the other two are dependent. Thus the simple pendulum has only one DOF.*

II. *Consider a ball dropped from a height H onto a rigid floor. The ball bounces on the floor and executes an up-down motion. The constraint on the particle's (ball's) motion is given by $z \geq 0$.*

III. *A box full of gas in which the molecules are restricted to the volume inside the box. The walls constrain the particles from moving out.*

IV. *Consider a rigid body composed of N point particles, where N is a very large number. In general we require $3N$ coordinates to describe the motion of the body of N particles. However, for every pair of particles there is an equation of constraint r_{ij}= constant and there are ${}^{N}C_2 = N(N-1)/2$ such equations i.e. one for every pair of particles. But all these equations of constraints are not independent of each other and we can show that finally a rigid body has 6 degrees of freedom i.e. requires 6 independent coordinates to describe its motion. Can you prove this?*

Mathematically constraints can be expressed as equations, inequalities, differentials or integrals and may be classified into 3 categories;

Holonomic

1. *Equalities of the coordinates $f_\lambda(x_i; t) = 0$ $(\lambda = 1, 2, ...k)$*

Non-Holonomic

2. *Inequalities like $f_\lambda(x_i; t) \geq 0$ $(\lambda = 1, 2, ...k)$*

3. *Linear non-integrable relations between differentials of the coordinates*

here f is a function of all coordinates and time and k is the number of constraints. The first kind of constraints are called holonomic constraints and the last two types are called non-holonomic constraints. The second kind of constraint can be made holonomic by stipulating that infinite forces (potentials) act on the particles in the regions where the constraint inequality is violated. Dealing with nonholonomic constraints of the third kind is non-trivial and we will not be discussing them extensively though will revisit them occasionally. Further if f is an explicit function of time t then those constraints are called a rheonomous constraints and if not then they are called scleronomous constraints.

As mentioned earlier, inclusion of constraints poses some challenges which need to be understood and overcome. Firstly, all the coordinates (e.g. x, y, z), are no longer independent and secondly, the forces of constraint, e.g., the tension in the string of the pendulum or the force exerted by the wall on the gas particles, are not known to us and must be determined as a part of the solution to the original problem. In cases with holonomic constraints $f_\lambda(x_i; t) = 0$, the first problem is solved by introducing generalized coordinates or those $3N - k$ number of coordinates. If there exists k constraint equations, they can be used to eliminate k of the $3N$ EOMs. Thus the system now has $3N - k$ independent coordinates and same number of DOF This elimination of the dependent coordinates can be done by introduction of $3N - k$ independent variables ($q_1, q_2, q_3, ...q_{3N-k}$) called the **generalised coordinates** *defined in terms of the old position variables as;*

Imposing constraints on the system is simply another way of stating that there are forces present in the problem that cannot be specified directly but are known rather in terms of their effect on the motion of the system.

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, ...q_{3N-k}, t) \tag{13}$$

where i runs from 1 to N. The above equations can then be inverted (transformed), subject to the equations of constraint to obtain $3N - k$ equations of the form;

$$q_i = q_i(r_1, r_2, r_3, ...r_N, t) \tag{14}$$

It is assumed that the backward and forwards transformation between r_i and q_i is always possible. The generalized coordinates are NOT necessarily orthogonal position coordinates, appearing in groups of 3 i.e. (x, y, z) or (r, θ, ϕ) or (ρ, ϕ, z). Many quantities can be used as generalized coordinates, with dimensions of charge, current, energy or angular momentum etc. Remember, that if the constraint is non-holonomic, the equations expressing the constraint cannot be used to eliminate the dependent coordinates. But there are alternative ways of addressing such problems.

Generalized coordinates do not necessarily have the dimension of length but can be any independent measurable variable specifying the state of the system

The second problem i.e. the forces of constraint, are not known beforehand. This can be addressed by developing a prescription of formulation of mechanics such that the need to know the constraint forces can be dispensed with and we work with only the known external forces. This can be effected by demanding that the net work done by the forces of constraint is zero!

What do you think about the work done by constraint forces in the case of the simple pendulum and for the ball bouncing on the rigid surface? Can you show that in both cases the work done by the constraint forces vanishes? Those interested in reading further about the topic can look up d'Alembert's Principle.

Consider the simple pendulum again oscillating in a vertical plane, we have 2 DOF (x, y) *and one equation of constraint* $x^2 + y^2$ = *constant, which tells us that there is effectively only 1 DOF. Now consider the coordinates* (r, θ) *where* $r^2 = x^2 + y^2$ *and* $\tan\theta = y/x$ *and the inverse transformation* $x = r\sin\theta$ *and* $y = r\cos\theta$*. Out of the new coordinates* (r, θ)*,* r *is a constant (=length of the string) and thus we require only 1 independent coordinate to describe the entire motion of the bob. Here,* θ *is the generalised coordinate for the case of the simple pendulum.*

Finally, given Newton's 2nd Law and its information content we understand that second order differential equations relating the second derivative of the coordinates with force allows us to determine the trajectory of the system. Thus to completely specify the state of the system we need to simultaneously specify **both** *its generalised coordinates* (q_i) *and their first derivatives i.e. the generalised velocities* $(\dot{q}_i)$ *at a instant of time. Only then is the state of the system completely determined. Note that specification of* (q_i) *alone is not sufficient, for without* $\dot{q}_i$ *at the same instant the system is free to evolve in infinitely many ways. The second derivative is only uniquely determined if the velocities are also known. Since both the* q_i*'s and the* $\dot{q}_i$*'s need to be specified to completely determine the system, in principle they are deemed to be independent quantities apriori. The final solution once we solve the problem will determine the time evolution of the* $q_i(t)$ *whence both* $\dot{q}_i$ *and* $\ddot{q}_i$ *may be determined at every instant.*

Homework

1. *Find 10 examples of the 3 different kinds of constraints discussed and their equations or inequalities.*

2. *Two point particles are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.*

Calculus of Variations and the Principle of Least Action

References:

1. *Marion and Thornton Chapters 6 and 7*
2. *Landau and Lifshitz Chapter 1*
3. *Simmons Chapter 7*

"But the real glory of science is that we can find a way of thinking such that the law is evident."
– *Feynman* (The Feynman Lectures in Physics, volume I)

Let us revisit the two problems we discussed at the beginning of the previous chapter. In the first problem you were required to find a route from Trivandrum to Kolkata, additionally now you are tasked to do it via the shortest route. Again, the shortest route would be through the gedanken tunnel connecting the 2 cities. Constrained to move on the surface of the earth, the shortest route would be along a geodesic connecting the 2 cities, which is only practical via air travel and not by any mode of land travel. For the later you'll be constrained to the existing road or rail network. Here, we encounter an optimisation problem (shortest route) subject to multiple constraints. In the second problem, on choosing the right cabin baggage the airline puts the constraint that length + breadth + height= 75 cm. A passenger has to follow that but would obviously want to maximize the volume of the bag to carry as many things as possible. So what should be the dimensions of the bag such that its volume (length × breadth × height) is maximum while conforming to the stipulated rule. Out of the infinite combination of the 3 variables, you can probably guess that the correct answer is a cube of side $a = \frac{75}{3}$*cm, which maximises the volume. But how we prove these?*

Having known calculus, you are familiar with taking derivatives and finding the extrema of functions. In physics extremisation principles applies to various problems e.g. finding equilibria, energy minimisation, entropy maximisation, ray optics etc. The Principle of Least Action *derives itself from the branch of mathematics known as the* **calculus of variations**. *The most well known application of the variational principle is perhaps Fermat's principle*[1] *that explains the laws of reflection and refraction of light, which says that light travels between two points along a path such that the time taken is the least. With ideas and applications dating back to antiquity*[2]*, modern studies into this powerful analytic technique was re-initiated by Euler in the* 18^{th} *century; after the development of calculus by Leibniz and*

[1] Pierre de Fermat (1607 -1665): French mathematician, noted for his pioneering work in analytic geometry, especially determining tangents to curves that led to the early development of differential calculus.

[2] Euclid's articulation of the variational principle explained the law of reflection i.e. equality of the angles of incidence and reflection

Newton.

The Calculus of Variations

We are all quite familiar with the problem of finding minima or maxima or extrema values of a function and the concepts of global and local maxima and minima. Systems in equilibrium are found to be at an extrema of its potential landscape; specifically a minima for stable equilibrium. In determining the above we search for a set of values of variables e.g. spatial coordinates for which the potential function is minimum, albeit within a specified region of space, i.e. allowed values of the variables.

Calculus of variations also deals with differential calculus of functions with a twist. Here, we search for an optimal function $Q(x)$, such that the integral, S of another function $f(Q(x), Q'(x); x)$, is extremal within a specified boundary (x_1, x_2).[3] *Mathematically it means, determine a function $Q(x)$ such that the integral of the function f, which is given to you, is an extremum.*

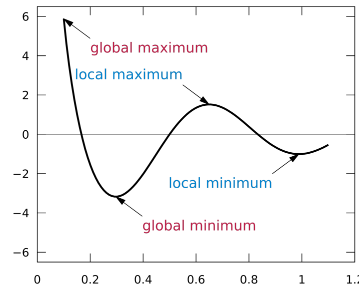


Figure 5: Local and global extrema of the function $\cos(3\pi x)/x$, $(0.1 \leq x \leq 1.1)$

[3] S is known as an integral functional, a generalisation of a function.
A function maps a set of input numbers to a resulting number by the transfer function, i.e. $z = f(x, y)$ maps an input (x, y) to an output number z. Similarly, a functional maps a function or a set of functions to an output set of numbers.

$$S = \int_{x_1}^{x_2} f(Q(x), Q'(x); x) dx \tag{15}$$

A few examples will help illustrate the matter. Say we dip a circular wire ring into a soap solution and bring it out. A soap film will span the wire ring, but our problem is to find the equation of the surface (Q). Our physical understanding says that the surface will be such that the total surface area (A), in reality the surface energy, of the soap film will be a minimum. What is the constraint here in this problem? A much easier problem deals with the finding the equation of a curve, joining 2 points on a plane, such that the length of the curve is minimum. In other words, prove that the shortest distance between 2 points on a plane is a straight line? which we know - intuitively. What about the shortest distance between 2 points on the earth i.e. surface of a sphere? Note: In the above equation x represents the independent variable, $Q(x)$ the dependent variable and $Q'(x) = dQ/dx$. The function $Q(x)$ is then varied until an extremal value of S is found. That is, if a function $Q(x)$ gives the integral a minimum value, then any neighbouring function, no matter how close to $Q(x)$, will increase S.

So how do we go about finding that optimal function $Q(x)$ such that S is an extremum? Lets vary $Q(x)$ in the following way, using a parameter α and an arbitrary function $\eta(x)$;

$$Q(\alpha, x) = Q(0, x) + \alpha\eta(x) \tag{16}$$

here $\eta(x)$ quantifies the variation to $Q(x)$

we only require that $\eta(x)$ has a continuous first derivative and is equal to zero at the endpoints i.e. $\eta(x_1)$ = 0 and $\eta(x_2)$ = 0. This is because value of the varied function $Q(\alpha, x)$ must be identical with $Q(x)$ at the endpoints of the path i.e. $Q_1 = Q(x_1)$ and $Q_2 = Q(x_2)$. With this varied function the *integral above becomes;*

$$S = \int_{x_1}^{x_2} f(Q(\alpha, x), Q'(\alpha, x); x) dx \tag{17}$$

In function theory the notion of a stationary value for a line integral corresponds to the vanishing of its first derivative i.e. S has the same value to within first order infinitesimals as that along all the varied paths. The necessary condition that the integral have a stationary/extremum value is that;

$$\frac{\partial S}{\partial \alpha}|_{\alpha \simeq 0} = 0 \tag{18}$$

for all functions $\eta(x)$. Differentiating S wrt α we get;

$$\frac{\partial S}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f(Q(\alpha, x), Q'(\alpha, x); x) dx \tag{19}$$

$$\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} (\frac{\partial f}{\partial Q} \frac{\partial Q}{\partial \alpha} + \frac{\partial f}{\partial Q'} \frac{\partial Q'}{\partial \alpha}) dx \tag{20}$$

Now;

$$\frac{\partial Q}{\partial \alpha} = \eta(x) \tag{21}$$

$$\frac{\partial Q'}{\partial \alpha} = \frac{d\eta}{dx} \tag{22}$$

The above differential equation becomes;

$$\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} (\frac{\partial f}{\partial Q} \eta(x) + \frac{\partial f}{\partial Q'} \frac{d\eta}{dx}) dx \tag{23}$$

The second term may be integrated by parts;

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial Q'} \frac{d\eta}{dx} dx = \frac{\partial f}{\partial Q'} \eta(x)|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} (\frac{\partial f}{\partial Q'}) \eta(x) dx \tag{24}$$

The first term vanishes since η is zero at the endpoints, then the differential of the integral becomes;

$$\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} (\frac{\partial f}{\partial Q} \eta(x) - \frac{d}{dx} (\frac{\partial f}{\partial Q'}) \eta(x) dx \tag{25}$$

$$= \int_{x_1}^{x_2} [\frac{\partial f}{\partial Q} - \frac{d}{dx} (\frac{\partial f}{\partial Q'})] \eta(x) dx \tag{26}$$

The functions Q and Q' with respect to which the derivatives f are taken are functions of α. Because $\partial S / \partial \alpha = 0$ must vanish for the extremum value and because $\eta(x)$ is an arbitrary function (subject to the conditions already stated), the integrand in equation must identically vanish for $\alpha = 0$, thus we get;

$$\frac{d}{dx} (\frac{\partial f}{\partial Q'}) - \frac{\partial f}{\partial Q} = 0 \tag{27}$$

where now Q and Q' are the original functions, independent of α. This is known as Euler's equation, derived by Euler in 1744.

Note that the differential quantity $\frac{\partial Q}{\partial \alpha} d\alpha = \eta d\alpha \equiv \delta Q$ represents infinitsimal variation from the right path (i.e. path for which $\delta S = 0$) at the point x and corresponds to the concept of "virtual displacement" that we will talk about separately. Similarly the variation of S about the right path may be written in terms of the parameter α as $\frac{\partial S}{\partial \alpha}|_{\alpha \simeq 0} d\alpha \equiv \delta S$. Further the equation 4.12 may be written as;

$$\delta S = \int_{x_1}^{x_2} [\frac{\partial f}{\partial Q} - \frac{d}{dx}(\frac{\partial f}{\partial Q'})] \delta Q dx \tag{28}$$

The notation δQ essentially denotes parametric variations between multiple alternate paths.

Problem 1

Task: *Find the equation of a curve joining 2 points (A - B) that lie on a plane.*
Condition: *(i) the length of the curve is a minimum.*
Total length of the curve;

$$length = \int_A^B dl = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + (\frac{dy}{dx})^2} dx \tag{29}$$

now $y[\equiv Q] = y(x) => dy = y'(x)dx$ and $f(y, y'; x) = \sqrt{1 + y'^2}$, the corresponding Euler's equation reads;

$$\frac{d}{dx}(\frac{\partial \sqrt{1+y'^2}}{\partial y'}) + \frac{\partial \sqrt{1+y'^2}}{\partial y} = 0 \tag{30}$$

$$\frac{d}{dx}(\frac{\partial \sqrt{1+y'^2}}{\partial y'}) = 0 \tag{31}$$

$$\frac{\partial \sqrt{1+y'^2}}{\partial y'} = A(constant) \tag{32}$$

$$\frac{y'}{\sqrt{1+y'^2}} = A \tag{33}$$

$$(1 - A^2) y'^2 = A^2 \tag{34}$$

$$y' = constant \implies y(x) = mx + C \tag{35}$$

which is the equation of a straight line - i.e. the shortest distance connecting any 2 points on a plane.

The brachistochrone problem is famous in the history of mathematics. The name derives from Greek meaning shortest time. It is the first variational problem that was formulated and solved by Johann Bernoulli, who posed it to readers of Acta Eruditorum in 1696 and led Bernoulli to the formal foundation of the calculus of variations. Four mathematicians responded with solutions: Isaac Newton, Jakob Bernoulli, Gottfried Leibniz and Guillaume de l'Hospital.
See the links Link 1 and Link 2

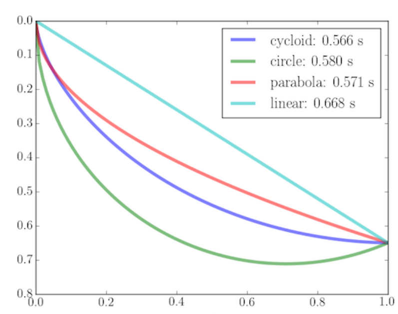


Figure 6: Time taken for travelling along various paths connecting A to B under action of gravity.

Problem 2

Task: *The Brachistrochrone: Determine the equation of the path taken by a particle on moving from one point to another under the action of gravity, on a vertical plane.*

Condition: *The time taken by a particle to move from the higher point (A) to the lower point (B), under the action of gravity is minimum. Time taken to travel from a point A to point B is given by;*

$$T_{AB} = \int_A^B \frac{ds}{v} \tag{36}$$

where s is the arc length and v is the speed given by $1/2mv^2 = mgy$ *(conservation of energy)* $\implies v = \sqrt{2gy}$*. Substituting back to equation 36 we get;*

$$\begin{aligned} T_{AB} &= \int_{P_1}^{P_2} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} \\ &= \int_{P_1}^{P_2} \frac{\sqrt{1 + dy'^2}}{\sqrt{2gy}} dx \end{aligned}$$

here the function in the integrand is $f(y, y') = \sqrt{1 + y'^2}/\sqrt{2gy}$*. The corresponding Euler's equation reads;*

Note that again $f(y, y')$ is not explicitly a function of x. In such cases it can be shown that $f - y' \frac{\partial f}{\partial y'} = C$, known as the Beltrami Identity.

$$\frac{d}{dx}\left(\frac{\partial(\sqrt{1 + y'^2}/\sqrt{2gy})}{\partial y'}\right) + \frac{\partial(\sqrt{1 + y'^2}/\sqrt{2gy})}{dy} = 0$$

which can be simplified to show that;

$$(1 + y'^2)y = 2k \ (constant)$$

The solution of this equation is given by the parametric equations,

A tautochrone or isochrone curve is a curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point on the curve. A cycloid is a isochrone and is traced by a point on a circle as it rolls along a straight line without slipping.

$$x = k(\theta - \sin\theta) \tag{37}$$

$$y = k(1 - \cos\theta) \tag{38}$$

which yields the equation of a cycloid.

Problem 3

Task: *Find the shortest path between two points lying on the surface of a sphere of radius r - the equation of a geodesic.*
Condition: *The 2 points lie on the surface of a sphere.*
Hint:The element of length on the surface of a sphere in the spherical polar coordinates is; $ds = r\sqrt{d\theta^2 + \sin^2\theta d\phi^2}$ *and the integral functional is,*

$$s = r\int_1^2 [\sqrt{(\frac{d\theta}{d\phi})^2 + \sin^2\theta}] d\phi \tag{39}$$

here $f = \sqrt{(\frac{d\theta}{d\phi})^2 + \sin^2\theta}$*. Find the equation of the geodesic and show that* $\cot\theta = b\sin(\phi - a)$*, where* a, b *are constants.*

Problem 4

Task: *Catenary: Find the shape of a rope/chain of uniform mass density λkg/m that hangs between two pegs that are at the same height, under the action of gravity. (Hint: the chain will hang such that its potential energy is minimised)*

Euler-Lagrange Equation (of the first kind)

When the Euler equation is applied to mechanical (physical) systems it is known as Euler-Lagrange equation. As discussed before, physical systems are described by n variables known as the generalised coordinates (q_i) and we are interested in obtaining the time dependence of these multiple generalised coordinates i.e. $q_i(t)$, with t as the independent variable. The instantaneous configuration of a system is described by specifying the $q_1 ... q_n$ coordinates that corresponds to a particular point in a n-dimensional hyperspace where the q's form the n coordinate axes - known as **configuration space**. *The system point moves in configuration space tracing out a curve - determined by the time dependence of each $q_i(t)$. Remember (i) the configuration space has no necessary connection with the physical three-dimensional space, just as the generalised coordinates are not necessarily position coordinates and (ii) the path traced by the system point in the configuration space has no resemblance to the path in space of any actual particle; each point on the path represents the entire system configuration at some given instant of time.*

Now consider the motion of mechanical systems, in which all forces (except the forces of constraint) are derivable from a scalar potential that may be a function of the coordinates, velocities, and time. Especially those where the potential function only of the position coordinates i.e. conservative. In such cases Hamilton's principle states that

"The motion of the system from time t_1 to time t_2 is such that the integral (called the **action** *or the action integral), S given below has a stationary value for the actual path of the motion".*

$$S \stackrel{def}{=} \int_{t_1}^{t_2} L(q_i, \dot{q}_i; t) dt \tag{40}$$

The above is the definition of S, an integral that yields a single number characterising a path. Hamilton's principle states that out of all possible paths by which the system point could travel from its starting point at time t_1 to end point at time t_2, in the configuration space it will actually travel along that path for which the value of the Action *i.e. the integral S is stationary or extremum, i.e. $\delta S = 0$.*

Note that the variation to the path at the end points is zero, i.e. the end points are fixed

$$\frac{\delta S}{\delta q(t)} = 0 \tag{41}$$

The action is the integral of a function $L(q_i, \dot{q}_i; t)$, called the Lagrangian, which is a function of the generalised coordinates (you can go on visualising them as spatial co-ordinates for ease of understanding but remember they need not necessarily be so), their time derivatives and time. In other words, any first-order change about the optimal path results in (at most) second-order changes in S. As we shall see Hamilton's principle replaces Newton's laws as the basic postulate in deriving the equations of motion (EOM) with several advantages. It is integral to the treatment of classical fields, plays an important role in quantum mechanics, quantum field theory and criticality theories.

Compare with the equations framed earlier and note the following replacements $x \to t$; $Q \to q_i$; $\dot{Q} \to \dot{q}_i$; $f \to L$. Now lets compute the quantity δS including the variations in the variables δq_i and $\delta \dot{q}_i$.

$$\delta S = \int_{t_1}^{t_2} \delta L \, dt \tag{42}$$

$$\delta S = \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i] dt \tag{43}$$

$$= \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i + \frac{\partial L}{\partial q_i} \delta q_i] dt = 0 \tag{44}$$

Again integrating the first term in the integrand by parts and requiring that all the variations δq_i's vanish at the end points we get;

$$\delta S = \int_{t_1}^{t_2} \sum_{i=1}^{n} [\frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_i}) + \frac{\partial L}{\partial q_i}] \delta q_i dt = 0 \tag{45}$$

Since all the q_i are independent and so are the variations $\delta\delta q_i$, the part in the square bracket must individually go to zero to ensure that δS is always zero, irrespective of the actual variations.

On the nature of "variations" δ

See section 6.7 of Marion and Thornton, **The δ Notation**

- *q_i are independent generalised coordinates and the variations δq_i are independent and arbitrary*

- *The variation of a time derivate is equal to the time derivative of the variation[4] i.e. $\delta \dot{q}_i = \frac{d}{dt}\delta q_i$. Thus the variations $\delta \dot{q}_i$ and δq_i are not mutually independent. In other words the variation $\delta \dot{q}_i$ means either the change induced in the tangent vector to $q_i(t)$ by the variation in the curve, or it means the time derivative of $\delta q_i(t)$*

[4] You can treat this as a postulate on the nature of variation, but do try to justify it.

- *Also note that taking variation δ of a functional is different from taking its derivative i.e. $\delta S \neq dS$.*

- *δq are akin to virtual displacements of the generalised coordinates and commensurate with the equations of constraints (they are necessarily holonomic)*

- *Finally, like δS, δq_is denote variation (and NOT differential) at every point on the path in configuration space i.e. at each instant of time. Note that there are no terms within the square brackets in equations 44 varying time itself. To reiterate, these variations are considered not with evolution of time but keeping time fixed!*

Following the derivation of Euler's equation in the previous section where f was a function of a single variable Q along with $\dot{Q}$ and t, the corresponding condition here is that L is a function of multiple independent generalized coordinates q_i, their time derivates and time. Note we have assumed that any constraints present are holonomic thus yielding the independent q_is. Following the treatment outlined in the previous section we can derive the Euler-Lagrange equations corresponding to each of the generalised coordinates as below. This follows directly from the requirement that the variations δq_i are independent.

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0........\{i = 1, 2, 3, ...n\}} \tag{46}$$

These are the Lagrange's equations of motions that follow from Hamilton's principle for systems with holonomic constraints. But what of the function $L(\dot{q}_i, q_i; t)$? As it turns out that $L = T - V$, where T and V are the kinetic and potential energies of the system being analysed. For the case of a particle moving under the influence of a conservative force field ($\vec{F} = -\nabla V(q_i)$). Equation 4.25 requires that the integral of $T - V$ is an extremum, not necessarily a minimum. But in almost all important applications in dynamics it turns out to be the minimum.

Problem 5

Consider the 1D harmonic oscillator i.e. a mass m attached to a massless spring of stiffness k and the equilibrium position of $x = 0$.

$$\begin{aligned}
L &= T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 && (47)\\
\frac{\partial L}{\partial \dot{x}} &= m\dot{x} && (48)\\
\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) &= m\ddot{x} && (49)\\
\frac{\partial L}{\partial x} &= kx && (50)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m\ddot{x} + kx = 0 && (51)
\end{aligned}$$

Equation 51 is the EOM you are familiar with and is readily obtained using Newton's second law. You will not be blamed for thinking that this was much ado about nothing! What could be done directly with the second law has been derived in a circuitous fashion. But there are definite advantages of what we are doing here so do bear with me.

Problem 6

Now consider the case of the simple pendulum. Using the plane polar coordinate system, such that θ is the angle made by the string of length l with the vertical downward direction, see figure 9. We can show that;

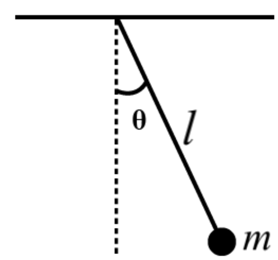


Figure 9: A bob of mass m hangs from a string of length l.

$$\begin{aligned}
T &= \frac{1}{2}ml^2\dot{\theta}^2 && (52)\\
V &= mgl(1 - \cos\theta) && (53)\\
L = T - V &= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) && (54)\\
\frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} && (55)\\
\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) &= ml^2\ddot{\theta} && (56)\\
\frac{\partial L}{\partial \theta} &= -mgl\sin\theta && (57)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \ddot{\theta} + \frac{g}{l}\sin\theta = 0 && (58)
\end{aligned}$$

again the EOM is the same as derived from Newton's law. Importantly, in ALL these examples force did not enter the calculations at all. Neither the force of gravity and its components, nor the forces of constraint i.e. tension, in the case of the pendulum. Here, the EOMs were obtained only by specifying the kinetic and potential energies.

Problem 7

Let's consider another example. Find the frequency of small oscillations of a simple pendulum placed in a railroad car that has a constant acceleration a in the x-direction. The kinetic and potential energies are given by;

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \qquad V = mgl(1 - \cos\theta) \tag{59}$$

Now can we show that the frequency of small oscillations is given by;

$$\omega^2 = \frac{\sqrt{a^2 + g^2}}{l} \tag{60}$$

page 243 Ex 7.6 in Marion and Thornton

Problem 8

Consider the motion of a bead that slides along a smooth wire bent in the shape of a vertical parabola $z = cr^2$. The bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω. Find the value of c. page 245 Ex 7.7 in Marion and Thornton

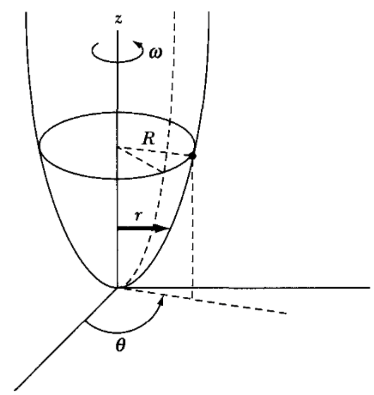


Figure 10: Bead on a parabolic wire rotating with angular speed ω.

Euler-Lagrange Equations (of the second kind) & Cyclic Coordinates

For all the problems solved above note that the Lagrangian of a system is not explicitly dependent on time i.e. $L = L(\dot{q}_i, q_i)$. Now;

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \quad + \quad \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i \tag{61}$$

$$\because \text{Lagrange's EOM is} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \quad = \quad 0 \tag{62}$$

$$\implies \frac{\partial L}{\partial q_i} \quad = \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) \tag{63}$$

Thus dL/dt can be written as;

$$\frac{dL}{dt} \quad = \quad \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\dot{q}_i \tag{64}$$

$$= \quad \sum_i \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) = \frac{d}{dt}\sum_i(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) \tag{65}$$

$$\implies \frac{dL}{dt} - \frac{d}{dt}\sum_i(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) \quad = \quad 0 \tag{66}$$

$$\implies \frac{d}{dt}\{\sum_i(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) - L\} \quad = \quad 0 \tag{67}$$

$$\implies \{\sum_i(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) - L\} \quad = \quad constant \tag{68}$$

This is the Euler-Lagrange equation of the second kind. Thus if the Lagrangian is not explicitly dependent on time then there exists a conserved quantity or a constant of motion called H, which may be a function of $\dot{q}$ and q, as given by the equation below.

In calculus of variations this is known as the Beltrami Identity, discovered in 1868 by Beltrami

$$\boxed{\text{If} \quad \frac{\partial L}{\partial t} = 0 \implies \{\sum_i (\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) - L\} = H \text{ (constant)}} \tag{69}$$

Now, a Lagrangian is a function of n independent generalised coordinates, their time derivates and time. Assume out of the n generalised coordinates, the j^{th} one does not appear explicitly in the Lagrangian function - such a coordinate is called a **cyclic coordinate**. *The corresponding EOM then reads;*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{70}$$

but the second term on the LHS $\frac{\partial L}{\partial q_j}$=0 anyway, $\because$ L is independent of q_j. Thus we can write;

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \tag{71}$$

$$\implies \frac{\partial L}{\partial \dot{q}_j} = p_j = \text{(constant)} \tag{72}$$

$p_j = \partial L/\partial \dot{q}_j$ is known as the **generalised- canonical-** *or* **conjugate-momemtum**. *Conjugate momentum $(\partial L/\partial \dot{q}_j)$ corresponding to each cyclic coordinate (q_j) is conserved and is a constant of motion.*

Problem 9

Revisit Problem 6, that of the simple pendulum where L is independent of time and calculate the conserved quantity. What is the physical identity of the quantity? ref: page 20 of handwritten notes

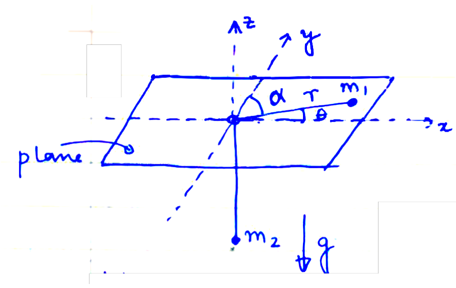


Figure 11: Two point masses m_1 and m_2 connected by a string.

Problem 10

Consider a horizontal surface (x, y) e.g. a frictionless tabletop, with a hole at the centre (0,0). Two particles of masses m_i and m_2 are attached by a massless, non-extendable string of length l. The string passes through the hole on the table such that m_1 rests on the table and m_2 hangs vertically below, under the action of gravity, as shown in fig: 11. Write the equations of constraints and obtain the equations of motion of the system. ref: page 13 of handwritten notes

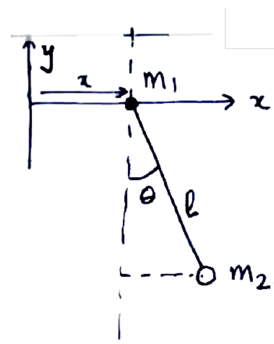


Figure 12: Simple pendulum with moving support

Problem 11

Consider a pendulum whose support can move along a horizontal line (fig: 12). Derive the EOM and the constants of motion if any. ref: page 17 of *handwritten notes*

The Principle of Least Action, the Lagrangian & the Equations of Motion

Historically, the concept of extremisation principles and their applications started with the field of optics. Hero of Alexandria (2^{nd}C) found that the law governing reflection of light could be obtained by asserting that a light ray, travelling from one point to another by reflection from a plane mirror, always takes the shortest possible path. However, this does not work for refraction of light, which allowed Fermat to conjecture that it is time that must be minimised in the process and not the length. The search for such principles is predicted on the notion that nature always minimizes certain vital quantities when a physical process takes place. The first application of a general minimum principle in mechanics in modern times is due to Maupertuis (1747), who asserted that dynamical motion takes place with minimum Action. Akin to Maupertuis definition of Action, here we have identified Action as a quantity with the dimensions of length $\times$ momentum or energy $\times$ time. The Least action principle forms the basis of mechanics, optics, quantum electrodynamics, optimal control theory etc and the maximum entropy principle (Clausiuss-Clayperon, Shannon) forms the base of thermodynamics and information theory.

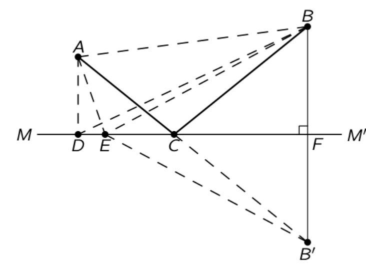


Figure 13: Principle of Least Time: reflection of light

In 1662, Fermat announced the proof of "Fermat's Principle". Claude Clerselier, a follower of Descartes and a leading Cartesian replied "*The principle you take as a basis for your proof, to wit, that nature always acts by the shortest and simplest ways, is only a moral principle, not a physical one—it is not and can not be the cause of any effect in nature.*"
- The Best of All Possible Worlds: Mathematics and Destiny, by Ivar Ekeland (University of Chicago Press)

1. *The integral called Action has the dimensions of energy $\times$ time or distance $\times$ momentum*

2. *The action integral assigns a value (number) to each of the infinite paths connecting the initial and final points that the system may follow.*

3. *The value of S though depends upon the physics of the system being considered. In classical mechanics the right path has the property that "nearby" paths do not change the value of S appreciably[5], and this is the essence of a variational principle. This is like identifying a it critical point condition among the various paths. Along that right path Newton's 2nd Law is satisfied.*

4. *The physics of the system is contained in the integrand, here given by the Lagrangian L*

5. *Validity of the principle of least action itself is independent of the reference frame used - inertial/non-inertial or otherwise*

6. *However, the functional form of the Lagrangian given as $L = T - V$ is valid only in inertial frames along with other restrictions discussed before*

7. *Unlike Newtonian dynamics, the Lagrangian method deals only with scalar quantities associated with the body (T and V), completely ignoring force. This extends the applicability of a least action principle i.e.*

[5] equivalent to saying $\delta S = 0$

Hamilton's principle to quantum mechanics, where forces are not known upfront.[6]

8. *Invariance of the Lagrangian: The Lagrangian is given by the difference between the kinetic and potential energies - hence is a scalar function and must be invariant with respect to coordinate transformations. Such transformations are not restricted to orthogonal coordinate systems but to generalised coordinates*

9. *The Euler Lagrange equations can be computed in any set of generalized coordinates and they are also guaranteed to be correct*

10. *Uniqueness of the Lagrangian: The Euler-Lagrange equation reads;*

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

where $L = T - V$, *now say we take an arbitrary function* $F(q_i, t)$ *and add its total derivative to L to create a new function* L'.

$$L' = L + \frac{dF}{dt} \tag{73}$$

$$\text{and} \quad \frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t} \tag{74}$$

Lets substitute L' *for L in the Euler-Lagrange equation and see if its satisfies the same;*

$$\tfrac{d}{dt}(\tfrac{\partial L'}{\partial \dot{q}_i}) - \tfrac{\partial L'}{\partial q_i} \stackrel{?}{=} 0 \tag{75}$$

$$\tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}_i}(L + dF/dt) - \tfrac{\partial}{\partial q_i}(L + dF/dt) \stackrel{?}{=} 0 \tag{76}$$

$$\tfrac{d}{dt}[\tfrac{\partial L}{\partial \dot{q}_i} + \tfrac{\partial}{\partial \dot{q}_i}(\textstyle\sum_j \tfrac{\partial F}{\partial q_j}\dot{q}_j + \tfrac{\partial F}{\partial t})] - \tfrac{\partial L}{\partial q_i} - \tfrac{\partial}{\partial q_i}(\textstyle\sum_j \tfrac{\partial F}{\partial q_j}\dot{q}_j + \tfrac{\partial F}{\partial t}) \stackrel{?}{=} 0 \tag{77}$$

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{q}_i} - \tfrac{\partial L}{\partial q_i} + \tfrac{d}{dt}\{\tfrac{\partial}{\partial \dot{q}_i}(\textstyle\sum_j \tfrac{\partial F}{\partial q_j}\dot{q}_j + \tfrac{\partial F}{\partial t})\} - \tfrac{\partial}{\partial q_i}(\textstyle\sum_j \tfrac{\partial F}{\partial q_j}\dot{q}_j + \tfrac{\partial F}{\partial t}) \stackrel{?}{=} 0 \tag{78}$$

since L is the original Lagrangian it satisfies the Euler-Lagrange equation, sum of the first 2 terms of equation 78 is equal to zero, and we are left to check if,

$$\tfrac{d}{dt}\{\tfrac{\partial}{\partial \dot{q}_i}(\textstyle\sum_j \tfrac{\partial F}{\partial q_j}\dot{q}_j + \tfrac{\partial F}{\partial t})\} - \tfrac{\partial}{\partial q_i}(\textstyle\sum_j \tfrac{\partial F}{\partial q_j}\dot{q}_j + \tfrac{\partial F}{\partial t}) \stackrel{?}{=} 0 \tag{79}$$

$$\tfrac{d}{dt}\{\tfrac{\partial F}{\partial q_i}\} - (\textstyle\sum_j \tfrac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j + \tfrac{\partial^2 F}{\partial q_i \partial t}) \stackrel{?}{=} 0 \tag{80}$$

$$\textstyle\sum_j \tfrac{\partial}{\partial q_j}\{\tfrac{\partial F}{\partial q_i}\}\dot{q}_j + \tfrac{\partial}{\partial t}\{\tfrac{\partial F}{\partial q_i}\} - (\textstyle\sum_j \tfrac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j + \tfrac{\partial^2 F}{\partial q_i \partial t}) \stackrel{?}{=} 0 \tag{81}$$

$$\textstyle\sum_j[\tfrac{\partial^2 F}{\partial q_j \partial q_i} - \tfrac{\partial^2 F}{\partial q_i \partial q_j}]\dot{q}_j + \tfrac{\partial^2 F}{\partial t \partial q_i} - \tfrac{\partial^2 F}{\partial q_i \partial t} \stackrel{?}{=} 0 \tag{82}$$

So when will this equation 79 be satisfied? Obviously when

$$\frac{\partial^2 F}{\partial q_j \partial q_i} = \frac{\partial^2 F}{\partial q_i \partial q_j} \qquad \text{and} \qquad \frac{\partial^2 F}{\partial t \partial q_i} = \frac{\partial^2 F}{\partial q_i \partial t} \tag{83}$$

[6] In Quantum Mechanics the number associated with each path yields a probability amplitude. Probability for the system to move from the initial state to the final via that path and is of the form $e^{\frac{i}{\hbar}S}$.

What are the conditions on the function F for that to happen and importantly what are the physical implications of such a demand. I'll leave it as an exercise for you to figure out. So we realise that the Lagrangian is no way unique and can be modified by addition of the time derivative of well behaved function F such that the Euler-Lagrange EOM remains unchanged.

What about the value of the new action integral $S' = \int_{t_1}^{t_2} L'(q_i, \dot{q}_i; t)dt$ *with the new Lagrangian* L'*?*

$$
\begin{aligned}
S' &= \int_{t_1}^{t_2} L'(q_i, \dot{q}_i; t)dt && (84)\\
&= \int_{t_1}^{t_2} [L(q_i, \dot{q}_i; t) + \frac{dF}{dt}]dt && (85)\\
&= \int_{t_1}^{t_2} L(q_i, \dot{q}_i; t)dt + \int_{t_1}^{t_2} \frac{dF}{dt}dt && (86)\\
&= \int_{t_1}^{t_2} L(q_i, \dot{q}_i; t)dt + \int_{t_1}^{t_2} dF && (87)\\
&= S + F(q_i, t_1) - F(q_i, t_2) && (88)
\end{aligned}
$$

The original action is modified by the addition of constant terms dependent solely on the end point coordinates. Thus the variation of the new action $\delta S' = \delta S + \delta F(q_i, t_1) - \delta F(q_i, t_2)$*. Since the variations are zero at the end points i.e.* $\delta q_i(t_1) = \delta q_i(t_2) = 0$*, it implies that* $\delta S' = \delta S = 0$ *i.e. the variation of* S' *is zero.*

11. *Ignorable or cyclic coordinates: Presence of cyclic coordinates implies the existence of conserved quantities*

12. *The EL equations are a system of n second-order differential equations for the curve* $q_i(t)$

Lagrange's Equations with Undetermined Multipliers

Recall our discussion on types of constraints (page 13), especially type 3 i.e. nonholonomic constraints that are non-integrable differentials of coordinates. Before we learn how to tackle or account for such constraints directly lets go back to the Euler's Equation. Consider a system with 2 independent variables Q_1 *and* Q_2 *and the function* $f(Q_1', Q_2', Q_1, Q_2, x)$

$$\frac{d}{dx}\frac{\partial f}{\partial Q_i'} - \frac{\partial f}{\partial Q_i} = 0 \qquad i = 1, 2 \tag{89}$$

Now in deriving the above equation by requiring the variation of the functional to be zero i.e. $\delta S = 0$ *we considered the varied paths;*

$$Q_i(\alpha, x) = Q_i(0, x) + \alpha\eta_i(x) \qquad i = 1, 2 \tag{90}$$

and then demanded that;

$$\frac{dS}{d\alpha}|_{\alpha=0} = \int_1^2 \{(\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1})\frac{\partial Q_1}{\partial \alpha} + (\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2})\frac{\partial Q_2}{\partial \alpha}\}dx = 0 \quad (91)$$

Note:

- *If the two Q_i's are independent and since their variations are arbitrary we could conclude that the individual prefactors of $\frac{\partial Q_i}{\partial \alpha}$ must be = 0 for the variation to be zero always.*
- *But if the Q_i's are NOT independent. There exists an constraint equation $g(Q_1, Q_2, x)$ = 0 implying that only one of the 2 coordinates are independent.*
- *In that case the variations η_1 $(= \frac{\partial Q_1}{\partial \alpha})$ and η_2 $(= \frac{\partial Q_2}{\partial \alpha})$, also are dependent*
- *If we include the variation of Q_1 and Q_2 in the constraint equation of g as $g(Q_1(\alpha, x), Q_2(\alpha, x), x)$.*
- *Now g is a function of α and thus we can calculate its variation with α as;*

$$\frac{dg}{d\alpha} = \frac{\partial g}{\partial Q_1}\frac{\partial Q_1}{\partial \alpha} + \frac{\partial g}{\partial Q_2}\frac{\partial Q_2}{\partial \alpha} = 0 \quad (92)$$

$$= \frac{\partial g}{\partial Q_1}\eta_1 + \frac{\partial g}{\partial Q_2}\eta_2 = 0 \quad (93)$$

$$\implies \eta_1 = -\{\frac{\partial g}{\partial Q_2} / \frac{\partial g}{\partial Q_1}\}\eta_2 \quad (94)$$

Now from equation 91 we have;

$$\frac{dS}{d\alpha}|_{\alpha=0} = \int_1^2 \{(\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1})\eta_1 + (\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2})\eta_2\}dx = 0 \quad (95)$$

But $\because$ η_1 and η_2 are inter-related, we can write

$$\frac{dS}{d\alpha} = \int_1^2 \{(\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1})\eta_1 - (\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2})(\frac{\partial g}{\partial Q_1} / \frac{\partial g}{\partial Q_2})\eta_1\}dx \quad (96)$$

$$\frac{dS}{d\alpha} = \int_1^2 \{(\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1}) - (\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2})(\frac{\partial g}{\partial Q_1} / \frac{\partial g}{\partial Q_2})\}\eta_1 dx \quad (97)$$

Now $\because$ η_1 is arbitrary we can demand that for the above to identically be equal to zero at α=0 the part in the curly brackets is always zero.

$$\implies (\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1})(\frac{\partial g}{\partial Q_1})^{-1} = (\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2})(\frac{\partial g}{\partial Q_2})^{-1} \quad (98)$$

Note:

- *that the LHS and RHS of the above equation are derivatives of 2 arbitrary functions f and g wrt to two different variables Q_1 and Q_2.*
- *The demand that they are equal can only be met only if they are both equal to a quantity independent of either Q_1 or Q_2*

Therefore we can write;

$$(\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1})\frac{\partial g}{\partial Q_1}^{-1} = (\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2})\frac{\partial g}{\partial Q_2}^{-1} = -\lambda(x) \tag{99}$$

Thus from the above we get 2 equations;

$$(\frac{d}{dx}\frac{\partial f}{\partial Q_1'} - \frac{\partial f}{\partial Q_1}) + \lambda\frac{\partial g}{\partial Q_1} = 0 \tag{100}$$

$$(\frac{d}{dx}\frac{\partial f}{\partial Q_2'} - \frac{\partial f}{\partial Q_2}) + \lambda\frac{\partial g}{\partial Q_2} = 0 \tag{101}$$

For the case of N variables Q_i and several (k) constraint equations $g_j(Q_i, x) = 0$ we get;

$$(\frac{d}{dx}\frac{\partial f}{\partial Q_i'} - \frac{\partial f}{\partial Q_i}) + \sum_{j=1}^{k}\lambda_j\frac{\partial g_j}{\partial Q_i} = 0 \tag{102}$$

Note:

- *there is a λ_i corresponding to each constraint and they are known as the Lagrange's undetermined multipliers*
- *its not the equations of constraints g_j that appear in the modified Euler's equation but only the differentials i.e. $\frac{\partial g}{\partial Q_j}$ - which may or may not be integrable.*
- *the scheme allows accommodation of non-holonomic constraints where the constraints may be expressed in terms of Q_i'.*

From the above we can generalise the application of Lagrange's undetermined multipliers to the case of the Euler-Lagrange equations of motion of N coordinates and k constraints, where $k < N$.

$$L = L(\dot{q}_i, q_i, t) \qquad i = 1, 2, ..., N \tag{103}$$

Note: Here q_i are not all independent. The k constraints are either given as;

$$g_j = g_j(\dot{q}_i, q_i, t) \qquad j = 1, 2, ..., k \tag{104}$$

or as;

$$\sum_i \frac{\partial g_j}{\partial q_i} dq_i = 0 \tag{105}$$

or a mixture of the both of the above. And then we get N modified EL equations given as;

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} + \sum_j \lambda_j(t)\frac{\partial g_j}{\partial q_i} = 0 \tag{106}$$

We have one such equation corresponding to each of the coordinates which are NOT independent of each other.
Note:

- *since in Hamilton's Principle the variations vanish at the end points i.e. holds time constant at the endpoints, addition of a term* $\frac{\partial g_j}{\partial t} dt$ *does not affect the final EOMs - see connection with non-uniqueness of Lagrangian discussed in previous section.*

- *what is the physical significance of* $\lambda(t)$*? Evidently in equation 106 the last term has the dimension of force and thus are related to the forces of constraints.*

- *problems where transformation to or identification of independent generalized coordinates is not possible this offers an alternative route to obtain the EOM*

- *applicable to cases where the constraint equations are expressed in terms of generalised velocities and may be non-integrable.*

Problem 12

Consider a disc of radius R rolling without slipping down an incline (angle α*). Derive the EOM discuss the forces of constraint. Note the constraint may be expressed as* $g(x, \theta) = x - R\theta = 0$ *or as* $\dot{x} = R\dot{\theta} \implies dx = Rd\theta$
ref: page 36 of handwritten notes, Ex 7.9 of Marion and Thornton and Page 49 Goldstein
Discuss the general case of a disc rolling down an incline but not necessarily in a straight line. What are the equations of constraints in such a case? What if the disc is allowed to tilt?

Problem 13

A particle of mass m rests on the top of a frictionless rigid hemisphere of radius R. Derive the EOM and the forces of constraint. ref: Ex 7.10 of Marion and Thornton

Problem 14

Revisit the Attwood's Machine using undetermined multipliers. Derive the EOM and the forces of constraint. ref: handwritten notes page 39

Special Cases of Frictional Forces
Frictional Force, $f_i = \alpha_i v_i^n$ $F = \sum_i^p \frac{\alpha_i v_i^{n+1}}{n+1}$
for n=1 we get $f_i = \alpha_i v_i$: $F = \sum_i^p \frac{\alpha_i v_i^2}{2}$, which is the Rayleigh dissipation function. Surface moving in contact with another, $df = \alpha v^n dA \implies F = \int \frac{\alpha v^{n+1}}{n+1} dA$

Velocity Dependent Potentials

Lecture Notes not available. Refer to the recorded lectures.

Symmetry Transformations and Noether's Theorem

Coordinate systems help us denote the "position" of a system in the 3D Euclidean space around us. And because there are various coordinate systems that are useful in different contexts we also need to understand the transformations from one system to the other. We have already encountered coordinate transformations like Cartesian to spherical polar or cylindrical etc. Lately we have also encountered generalised coordinates which may be more complex than mere spatial variables. There are various kinds and classes of transformations that provide the link between the old and new coordinate systems, and the information they carry.

- **Coordinate Transformations***:*

  Suppose we change our coordinates from $(q, \dot{q}, t) \longleftrightarrow (q', \dot{q}', t)$. The original and the new Lagrangians are related by;

  $$L'(q', \dot{q}', t) = L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t)$$

  Obviously the functional form of the Lagrangians in terms of the new coordinates may be different, as may be the EOMs derived, but they will be equivalent since the physics and the resulting evolution will be the same.

  Any such transformation (change of coordinates) may be viewed in 2 ways (i) active and (ii) passive. Active transformation: The change in coordinates denote a real movement of the system point(s) in the configuration space but the coordinate axes themselves remain the same as original. Passive transformation: A new coordinate system has been invoked thus relabelling all points in configuration space. Transformations such as translation, rotation, reflection all may be viewed as active or passive transformations and both pictures are equivalent.

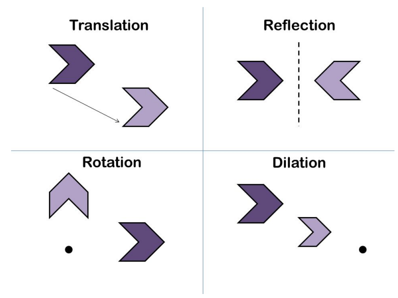


Figure 14: Some simple transformations

- **Continuous Transformations:** *These are transformations that can be written as a function of a continuous parameter(s) ϵ_i, such that $q'(t) = Q(\epsilon_i, t)$ with $Q(0, t) = q(t)$. The rotation and translation transformations above are examples of continuous transformations, with the ϵ_i being the rotation angles θ or the translation distances s, respectively.*

- **Symmetry Transformations:** *Now, let's consider transformations such that the Lagrangian is invariant under the transformation;*

  $$L' = L(q', \dot{q}', t)$$

  if the above holds, that the system is symmetric under the transformation or that the transformation is a symmetry transformation of the system. Here we require that in spite of relabelling of the coordinates, the form of

Not all transformations are symmetry transformations. Consider a simple Lagrangian, $L = \frac{m}{2}\dot{\vec{r}}^2 - V(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. You can easily show that L is invariant under reflection and rotation.

1. reflection: $x' = -x, y' = y, z' = z$
2. rotation: $x' = x\cos\theta - y\sin\theta, y' = x\sin\theta + y\cos\theta, z' = z,$

   but what about,
3. translation $x' = x + a, y' = y, z' = z$?

Here $L'(x', y', z') \neq L(x', y'z')$, the Lagrangian is NOT invariant under the transformation.

the **original** *Lagrangian still holds. Note: Any coordinate transformation will result in a perfectly valid transformed Lagrangian that will yield a trajectory equivalent to the solution from the original Lagrangian, even though the Lagrangians are different. Under a symmetry transformation, not only are the trajectories equivalent, but the Lagrangians are the same.*

Now consider a **continuous symmetry** *transformation i.e.* $L' = L(q', \dot{q}', t)$ *where* $q' = Q(\epsilon, t)$*. If* $Q(\epsilon, t)$ *is a symmetry transformation of the L, then* L' *does not depend on* ϵ*:*

$$\frac{d}{d\epsilon}L(Q, \dot{Q}, t) = \frac{d}{d\epsilon}L(q, \dot{q}, t) = 0 \tag{107}$$

The above statement is counter intuitive since $L(Q(\epsilon, t), \dot{Q}(\epsilon, t), t)$ *is explicitly dependent on* ϵ*. To understand the implication of the above consider the differential;*

$$\frac{d}{d\epsilon}L(Q, \dot{Q}, t) = \frac{\partial L}{\partial Q}\frac{dQ}{d\epsilon} + \frac{\partial L}{\partial \dot{Q}}\frac{d\dot{Q}}{d\epsilon} = 0$$

Now since L satisfies the Euler Lagrange EOM $\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} - \frac{\partial L}{\partial Q} = 0$*, the above equation becomes;*

$$\begin{aligned} \frac{d}{dt}(\frac{\partial L}{\partial \dot{Q}})\frac{dQ}{d\epsilon} + \frac{\partial L}{\partial \dot{Q}}\frac{d\dot{Q}}{d\epsilon} = 0 \\ \frac{d}{dt}(\frac{\partial L}{\partial \dot{Q}}\frac{dQ}{d\epsilon}) = 0 \end{aligned}$$

The first term in the parenthesis is the generalised momemtum corresponding to the generalised coordinate. Thus the above implies;

$$\begin{aligned} \frac{d}{dt}(\frac{\partial L}{\partial \dot{Q}}\frac{dQ}{d\epsilon}) &= \frac{d}{dt}(p\frac{dQ}{d\epsilon}) = 0 \\ \implies p\frac{dQ}{d\epsilon} &= constant \end{aligned}$$

The value of the constant may be evaluated for the case $\epsilon = 0$ *i.e. constant* $= p\frac{dQ}{d\epsilon}|_{\epsilon=0}$*. In general, if a Lagrangian of N coordinates possesses a set of M continuous symmetry transformations parameterized by parameters* ϵ_i*, then there are M conserved quantities associated with the transformations given by;*

$$\sum_{k=1}^{N} p\frac{dQ_k}{d\epsilon_i}\bigg|_{\epsilon_i=0} \tag{108}$$

Problem 15

A point particle moves in a gravitational field acting along $-z$ *direction and consider the continuous transformation of translation by* $\vec{s}$*. The original Lagrangian is,*

$$L(x, y, z) = \frac{m}{2}\dot{\vec{r}}^2 - mgz$$

under the transformation $x' = x + s_x, y' = y + s_y, z' = z + s_z$ the new Lagrangian is

$$L(x', y', z') = \frac{m}{2}\dot{\vec{r}}^2 - mgz' + mgs_z$$

The new Lagrangian invariant if $s_z = 0$ i.e., if the translation $\vec{s}$ is perpendicular to the direction of gravity. So the above transformation will be a symmetry transformation if $x' = x + s_x, y' = y + s_y$, and $z' = z$. Since the transformation has 2 continuous symmetry transformations denoted by the parameters s_x and s_y there will be two constants or conserved quantities;

$$p_x \frac{dx'}{ds_x}|_{s_x=0} + p_y \frac{dy'}{ds_x}|_{s_x=0} = p_x$$
$$p_x \frac{dx'}{ds_y}|_{s_y=0} + p_y \frac{dy'}{ds_y}|_{s_x=0} = p_y$$

which are the conventional mechanical momenta along the x and y directions.

Problem 16

Consider the motion of a point particle in a spherically symmetric potential where the Lagrangian is given by $L = \frac{m}{2}\dot{\vec{r}}^2 - V(r)$ under the continuous symmetry transformation of rotation about an **arbitrary** *z axis $x' = x\cos\theta - y\sin\theta, y' = x\sin\theta + y\cos\theta, z' = z$. Corresponding to the single parameter θ there will be one constant or conserved quantity,*

$$p_x \frac{dx'}{d\theta}\bigg|_{\theta=0} + p_y \frac{dy'}{d\theta}\bigg|_{\theta=0} = constant$$
$$p_x(x\sin 0 - y\cos 0) + p_y(x\cos 0 - y\sin 0) = constant$$
$$\implies xp_y - yp_x = constant = l_z$$

which is the z component of angular momentum. Similarly, rotations about the x and y axes would have yielded conserved quantities $l_x = yp_z - zp_y$ and $l_y = zp_x - xp_z$.

Noether's Theorem

Equation 108 leads to Noether's theorem, who proved that every differentiable (thus continuous) symmetry of the Action of a physical system with conservative forces has a corresponding conserved quantity and satisfies a conservation law. However, the notion of invariance of the Lagrangian may be extended to include t in addition to generalised coordinates. Therefore, in general if the Lagrangian is invariant under small perturbations of the time variable t and the generalized coordinates q we can write;

$$t \to t' = t + \delta t$$
$$\mathbf{q} \to \mathbf{q}' = \mathbf{q} + \delta\mathbf{q}$$

Figure 15: Emmy Noether

Amalie Emmy Noether (23 March 1882 – 14 April 1935) was a German mathematician who made important contributions to abstract algebra. Importantly in physics, Noether's theorem explains the connection between symmetry and conservation laws. www.youtube.com/watch?v=tNNyAyMRsgE *where the perturbations δt and δq are* **small** *and under a continuous transformation may be written as*

$$\begin{aligned} \delta t &= \epsilon T \\ \delta \mathbf{q} &= \epsilon \mathbf{Q} \end{aligned}$$

where ϵ is the infinitesimal transformation parameter coefficient corresponding to each generator T of time evolution, and the generator $\mathbf{Q}$ of the generalized coordinates. For translations, $\mathbf{Q}$ is a constant with unit of length; for rotations, it is an expression linear in the components of $\mathbf{q}$, and the parameter ϵ an angle[7]*. Using these definitions, Noether showed that for M continuous transformations defined by a set of parameters ϵ_i there will be a conserved quantity corresponding to each transformation given by;*

[7] Hopefully, you can easily visualize multiple rotations about different axes, thus having multiple ϵ and corresponding generators

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L\right) T_i - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_i \qquad (109)$$

Time invariance: *a Lagrangian that does not depend on time ($\frac{\partial L}{\partial t} = 0$) or is invariant under time translation. Here, T = 1 and Q = 0; the corresponding conserved quantity is;*

$$H = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - L$$

identified as the total energy of the system. Homogeneity of time leads to conservation of energy.

Translational invariance: *a Lagrangian that does not depend on a particular coordinate (q_k is cyclic or ignorable) or is invariant under changes $q_k \to q_k + \delta q_k$. Here, T = 0, and $Q_k = 1$ and the conserved quantity is the corresponding conjugate momentum,*

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Homogeneity of space leads to conservation of linear momentum. **Rotational invariance:** *a Lagrangian does not depend on the absolute orientation of the physical system in space or the Lagrangian does not change under small rotation by an angle $\delta\theta$ about an arbitrary axis given by $\hat{n}$. Such a rotation transforms the Cartesian coordinates by $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \delta\theta(\hat{n} \times \mathbf{r})$. Here, T = 0 and $\epsilon = \delta\theta$ and $\mathbf{Q} = \hat{n} \times \mathbf{r}$. Then Noether's theorem states that the following quantity is conserved,*

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_r = \mathbf{p} \cdot (\mathbf{n} \times \mathbf{r}) = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{p}) = \mathbf{n} \cdot \mathbf{L}$$

In other words, the component of the angular momentum $\mathbf{L}$ along the axis of rotation $\hat{n}$ is conserved. If the system is insensitive to any rotation ($\hat{n}$ is arbitrary) then every component of $\mathbf{L}$ is conserved; or the total angular momentum is conserved. Isotropy of space leads to conservation of angular momentum.

Problem 17

Revisit the damped harmonic oscillator (Homework Problem 19), for which the correct EOM can be derived from a Lagrangian of the form;

$$L = e^{\gamma t}(\dot{x}^2 - \omega^2 x^2)$$

Prove that this L is invariant under the transformation $t' = t + \epsilon$ *and* $x' = x - \epsilon\frac{\gamma x}{2}$*. Remember that* ϵ *is small and* $\dot{x}' = \dot{x} - \epsilon\frac{\gamma\dot{x}}{2} = \dot{x}(1 - \epsilon\frac{\gamma}{2})$*. Show that the invariance of L leads to a conserved quantity given by;*

$$p_x(\dot{x} - \frac{\gamma x}{2}) - L = constant$$

where $p_x = \frac{\partial L}{\partial \dot{x}}$.

Homework

Principle of Least Action and Euler's Equation

1. *Find the extremals for the integral* $J = \int_{x_1}^{x_2} f(y', y, x)dx$ *if the integrand is given by (i)*y'^2 *(ii)* $y^2 + y'^2$ *(iii)* $y/(1 + y'^2)$ *(iv)* $y'/(1 + y^2)$

2. *Find the shape of the curve that encloses the largest area, such that the length of the curve (perimeter) is a given constant.*

3. *Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.*

4. *Consider light passing from one medium with index of refraction* n_1*, into another medium with index of refraction* n_2*. Use Fermat's principle to minimize time, and derive the law of refraction:* $n_1 \sin\theta_1 = n_2 \sin\theta_2$*. Assume the interface between the media is a flat.*

5. *Find the ratio of the radius R to the height H ot a right-circular cylinder of fixed volume V that minimizes the surface area A.*

6. *Find the shortest path between the (x, y, z) points (0,—1,0) and (0,1,0) on the conical surface* $z = 1 - \sqrt{x^2 + y^2}$*. What is the length of the path? Note: this is the shortest mountain path around a volcano.*

7. *The corners of a rectangle lie on the ellipse* $(x/a)^2 + (y/b)^2 = 1$*. (a) Where should the corners be located in order to maximize the area of the rectangle? (b) What fraction of the area of the ellipse is covered by the rectangle with maximum area?*

Lagrange's EOM

8. *Consider the projectile motion of a particle under the action of gravity, in a vertical plane i.e. 2D. Find the EOM in both Cartesian and polar coordinates.*

9. *A particle of mass m is constrained to move on the inside surface of a frictionless cone of half-angle* α*, under gravity. Determine a set of generalized coordinates and determine the constraints. Find Lagrange's EOM.* *page 240 Ex 7.4 in Marion and Thornton*

10. *The point of support of a simple pendulum of length l moves on a massless rim of radius a rotating with constant angular velocity* ω*. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass m in terms of the plane polar coordinates.* *page 242 Ex 7.5 in Marion and Thornton*

11. *Consider an extension of classical mechanics where the Lagrangian is of the form* $L(q, \dot{q}, \ddot{q}; t)$ *for the generalized coordinates q. Make use of*

Hamilton's principle such that any variation for both q and $\dot{q}$ vanishes at the end points. Show that the EL EOM will then be given by;

$$\frac{d^2}{dt^2}(\frac{\partial L}{\partial \ddot{q}}) - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) + \frac{\partial L}{\partial q} = 0$$

Now apply the above to the Lagrangian below and make any comments that you can.

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$$

12. *The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where k and b are constants. Find the equations of motion.*

Time dependent problems

13. *Consider a simple plane pendulum consisting of a mass m attached to a string of length I. After the pendulum is set into motion, the length of the string is shortened at a constant rate $dl/dt = -\alpha(constant)$. The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.*

14. *A particle of mass m moves in one dimension under the influence of a force*

$$F(x, t) = \frac{k}{x^2}e^{-t/\tau}$$

where k and t are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

15. *The point of support of a simple pendulum of mass m and length b is driven horizontally by $x = a\sin\omega t$. Find the pendulum's equation of motion.*

Lagrange's Undetermined Multipliers

16. *Use the method of Lagrange undetermined multipliers to find the tensions in both strings of the double Atwood machine, solved as Problem 14 in solved examples.*

17. *A particle of mass m starts at rest from the top of a inverted hemisphere, of radius R. Find the force of constraint and calculate the angle at which the particle leaves the hemisphere.*

18. *A particle of mass m is suspended by a massless spring of length l. It hangs, without initial motion, and is struck by an impulsive horizontal blow, which introduces an angular velocity ω. If ω is sufficiently small, it is obvious that the mass moves as a simple pendulum. If ω is*

sufficiently large, the mass will rotate about the support. Use a Lagrange multiplier to determine the conditions under which the string becomes slack at some point in the motion. Acceleration due to gravity g acts vertically downwards.

Velocity Dependent Potentials - Friction & Dissipation

19. *In certain situations, particularly one-dimensional systems, it is possible to incorporate frictional effects without introducing the dissipation function. As an example, find the equations of motion for the Lagrangian* $L = (\dot{x}^2 - \omega^2 x^2)e^{\gamma t}$ *How would you describe the system? Are there any constants of motion? Suppose a point transformation is made of from* $x \to s$ *the form* $s = xe^{\gamma t}$ *What is the effective Lagrangian in terms of s? Find the EOM for s. What do these results say about the conserved quantities for the system?*

20. *In removing a tightly fitting cylinder from inside another, why do we have to twist one with respect to the other as they are pulled apart - similar to removing a cork from a bottle.*

21. *A flat circular disk of radius r is in contact, with a plane surface coated with oil. Assuming the oil exerts a uniform viscous drag on every element of area of the disk, show that the frictional forces coresponding to x,y, are* $F_x = -Aa\dot{x}$, $F_y = -Aa\dot{y}$ *and* $F_\theta = -Aar^2\dot{\theta}/2$ *where x, y locate the center of the disk and θ its angular position.* $A = \pi r^2$ *and a is the viscous force per unit area per unit velocity. Note that each force depends only on the corresponding velocity.*

Applications to Electrical Circuits

22. *Consider a series LCR circuit connected to a dc voltage source of emf* E_o. *Using Kirchoff's 2nd law write the dynamical equation for current/charge flowing across the circuit. Now assuming charge (Q) as the generalised coordinate write the Lagrangian of the system as the difference between the kinetic* ($L\dot{Q}^2/2$) *and potential* ($Q^2/2C$) *energy terms. Show that effect of the resistor (R) can be dealt with by incorporating a dissipative Rayleigh function.*

23. *A form of the Wheatstone bridge has, in addition to the usual 4 resistances, an inductance in one arm and a capacitance in the opposite arm. Set up the Lagrangian (L) and the Rayleigh dissipation function for the unbalanced bridge, with the charges in the elements as generalised coordinates. Using the Kirchhoff's Laws as constraints on the currents, obtain the Lagrange equations of motion, and show that eliminating the undetermined multipliers reduces these to the usual network equations.*

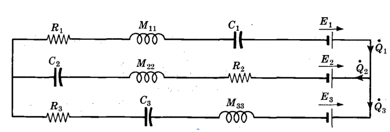


Figure 16: Electrical Circuit

24. *Consider the circuit given in Fig 16 and write down the Lagrangian of the circuit. M_{ii} are the self inductance of the inductor, assume M_{ij} denotes the mutual inductance between 2 inductors.*

25. *In Fig. 17(a) a sphere of mass m is suspended in a viscous liquid from a coil spring l_o = unstretched length of spring, y_o = elongation of spring with m at rest, y = general displacement from rest position. We assume that the only effect of the liquid is to exert a viscous drag $-ay$. Fig. 17 (b) represents a simple series electrical circuit. Lagrangian functions for (a) and (b) respectively are $L_{Me} = m\dot{y}^2/2 - k(y + y_o)2/2 + mgy$ and $L_{EI} = MQ^2/2 - Q^2/C + EQ$. Show that the systems are equivalent by deriving their dynamical equations.*

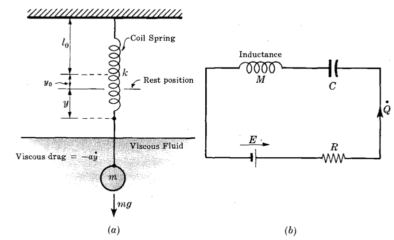


Figure 17: Mechanical - Electrical equivalence

26. *The inside half-cylinder A, Fig. 18, supported in a vertical position by a thin elastic rod (torsional constant k) fastened along its axis at O, can rotate within B. Assuming that the capacity of this variable condenser is given by $C = C_O(1 - \theta/\pi)$ and that the rod is undistorted for $\theta = \theta_1$, obtain the Lagrangian and the dynamical equations of the circuit.*

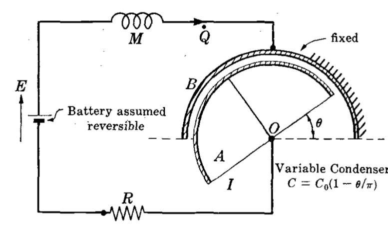


Figure 18: Variable capacitor

27. *Referring to previous problem, Fig. 18, show that equilibrium values of θ and Q are given by $\theta_0 = \theta_1 - C_0E^2/2\pi k$, $Q_0 = C_O(1 - \theta_0/\pi)E$. It can be seen from the physics involved that when the condenser is charged, $\theta_1 \geq 0$. Find equations of motion which determine the oscillations of θ and Q about equilibrium values.*

28. *Each plate of the variable condenser in Fig. 19 , is free to move along a line ab without rotation, under the action of a spring and the electric field between them. Derive the Lagrangian and EOM.*
    $L = \frac{1}{2}(m_1\dot{x_1}^2 + m_2\dot{x_2}^2 + M\dot{Q}^2) - \frac{1}{2}(k_1x_1^2 + k_2x_2^2) + EQ - \frac{1}{2}Q^2(s - x_1 - x_2)/A$

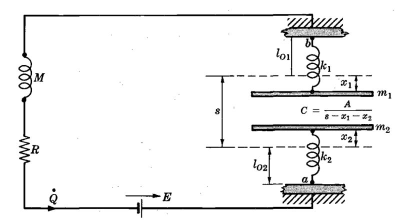


Figure 19: Circuit with a resistor R, inductance M and variable capacitor C of area A.

Symmetry

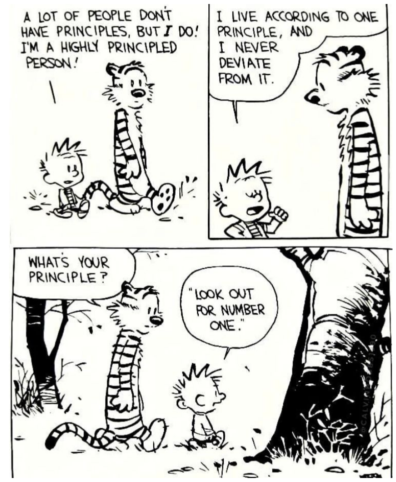
29. *Which of the following forces would violate mirror symmetry?*
    (a) $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$
    (b) $\vec{F} = q\vec{v} \times \vec{B} + m\vec{g}$
    (c) $\vec{F} = q\alpha\vec{E} \times \vec{B}$
    In the above, $\vec{E}$ *is the electric field,* $\vec{B}$ *is the magnetic fields,* $\vec{g}$ *is the acceleration due to gravity, and* α *is a constant.*

30. *Prove Noether's Theorem. A continuous symmetry transformation yields a corresponding constant of motion given by;*

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L\right) T_i - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_i \tag{110}$$

Advanced Problems

31. *Show that if $K(\dot{x}, x)$ be any constant of motion, then the Lagrangian $L = \dot{x} \int K/\dot{x}^2 d\dot{x}$ leads to the EOM. Acta Physica Austrtriaca 51, 293, 1979. Can you determine the function K for the damped harmonic oscillator where the EOM is given by $\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0$*



32. *Is it really the Principle of Least Action?*
Physical trajectories $x(t)$ (obeying Newton's 2nd law) are critical solutions of the functional $S(x) = \int L dt$, where $L = T - V$. The variational principle is called the principle of least action since it is assumed that S is minimized by the curve satisfying the EOM, but this is not necessary condition for $\delta S = 0$. Akin to ordinary calculus, the existence of a critical point indicates the existence of either a local maximum or a minimum or a saddle point. But what can you say about the nature of the right path $x(t)$)? Expand the action to second order in variations. $\because x(t)$ is the right path,

$$S(x + \delta x) = S(x) + \delta^2 S + \mathcal{O}(\delta x^3),$$

where $\delta^2 S$ is called the second variation of the action about the critical path $x(t)$. For a particle moving in 1-d, in a potential $V(x)$ show that;

$$\delta^2 S = \int_0^T \frac{1}{2}\left(m(\delta\dot{x})^2 - \frac{\partial^2 V}{\partial x^2}\bigg|_{x(t)} (\delta x)^2 \right) dt$$

Now is $\delta^2 S > 0$? or $\delta^2 S < 0$? or $\delta^2 S = 0$? Can you argue that taking the time interval T sufficiently small the second term in the integrand will be much smaller than the first term thus making $\delta^2 S > 0$, $\therefore$ the action is minimized on the right path?

33. *Study Euler's Theorem that states that if $f(x_i)$ is a homogeneous function of x_i (i.e. $f(\alpha x) = \alpha f(x_i)$) of degree n then we can show that;*

$$\sum_i x_i \frac{\partial f}{\partial x_i} = nf$$

Note the connection to the Kinetic Energy function which is a homogeneous function of $\dot{x}_i$ of degree 2. Thus;

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

Pringle's Potato Chips

34. *The shape of Pringle's potato chips is a hyperbolic paraboloid.*
The shape has many useful applications in the physical world from stackable potato chips to structural building elements. Draw the shortest path connecting any two points lying on the edge of such a chip? (3 points are marked in red for reference.) What about the equation of such a path?

Central Force Motion

Reference Frames

Let S and S′ be two reference frames. Let $\vec{R}$ be the position vector of S′ with respect to frame S. $\vec{r}$ denotes the position vector of a point particle with respect to frame S and $\vec{r'}$ denotes the position vector with respect to S′. The position vectors are related by

$$\vec{r} = \vec{r'} + \vec{R} \tag{111}$$

And the relative velocity between the two reference frames is given by

$$\vec{V} = \frac{d\vec{R}}{dt} \tag{112}$$

If the relative velocity between the two reference frames ($\vec{V}$) is constant then, the relative acceleration between the two reference frames is zero, i.e. $\vec{A} = \frac{d\vec{V}}{dt} = 0$. Then the reference frames S and S′ are called relatively inertial reference frames.

The velocity of the point particle will be different when measured from the two reference frames. If $\vec{v}$ denotes the velocity vector of the point particle with respect to frame S and $\vec{v'}$ denotes the velocity vector with respect to S′, then the velocity vectors are related by

$$\vec{v} = \vec{v'} + \vec{V} \tag{113}$$

The centre of mass (CM) is defined as the point where the mass-weighted position vectors (moments) relative to the point sum to zero.

The Centre of Mass Reference Frame

Consider a collection of n point particles with position vectors given by $\vec{r}_i$. In the S frame the CM is the mean location of a mass distribution in space (see fig. 20) and is given by;

$$\vec{R}_{CM} = \frac{\sum_1^n m_i \vec{r}_i}{M} \tag{114}$$

where $M = \sum_{i=1}^n m_i$. Similarly the velocity of the CM is given by;

$$\vec{V}_{CM} = \frac{\sum_1^n m_i \vec{v}_i}{M} = \vec{p}_{total} \tag{115}$$

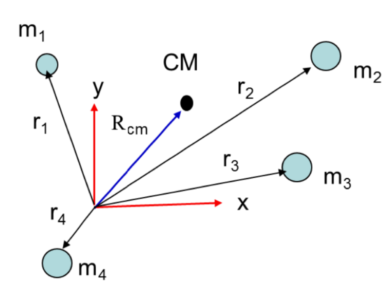


Figure 20: Collection of point particles and their CM.

Now consider the frame S′ attached to the CM. The velocity of each particle wrt the CM frame (S′) is $\vec{v}'_i = \vec{v}_i - \vec{V}_{CM}$ *and the total momentum in S′ is;*

$$\vec{p'}_{total} = \sum \vec{p}'_i = \sum_{i=1}^{n} m_i \vec{v}'_i = \sum_{1}^{n} m_i(\vec{v}_i - \vec{V}_{CM}) \tag{116}$$

$$= \sum_{i=1}^{n} m_i \vec{v}_i - \sum_{i=1}^{n} m_i \frac{\sum_{j=1}^{n} m_j \vec{v}_j}{M} \tag{117}$$

$$= \sum_{i=1}^{n} m_i \vec{v}_i - \sum_{j=1}^{n} m_j \vec{v}_j = 0 \tag{118}$$

The total momentum of a system of particles in the CM frame is equal to zero.

Additionally, we can show that the total energy of the system is a minimum in the CM frame, compared to all other inertial reference frames.

From here-on we deal with 2 inertial frames:

- *The Laboratory frame (S): this is the frame where measurements are actually made*
- *The centre of mass frame (S′): this is the frame where the centre of mass of the system is at rest and with respect to which the total momentum of the system is zero*

A system of 2 particles (interacting)

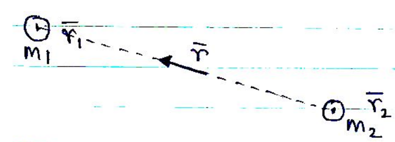


Figure 21: Two body system, where $\vec{r} = \vec{r}_1 - \vec{r}_2$

Consider a closed system of 2 point particles that are interacting with each other through mutual interactive forces (e.g. gravitational forces) directed along the line connecting them (i.e. central force). We require 6 position coordinates defining $\vec{r}_1$ *and* $\vec{r}_2$ *in S frame to describe the system, with the Lagrangian given by;*

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - V(\vec{r}) \tag{119}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ *and* $V(\vec{r})$ *scalar potential defining the interaction force between the particles. We can easily show that in the absence of external forces (closed system) the* $\ddot{\vec{R}}_{CM} = 0$*, i.e. the CM frame is an inertial frame and moves with a constant velocity. Similarly, it is easily shown that the kinetic energy in S is equal to the kinetic energy in S′ + the kinetic energy of the CM, using the fact that* $\vec{p'}_{total} = 0$*.*

In the CM frame (S′), sum of the mass moments is zero.

$$\sum_{1}^{n} m_i \vec{r}'_i = 0 \tag{120}$$

and the position vector of the Center of Mass (CM) is given by,

$$\vec{R} = \frac{\Sigma m_i \vec{r}_i}{M} \tag{121}$$

were, $M = \Sigma m_i$. *We also know that,*

$$\frac{d\vec{p}}{dt} = 0 \tag{122}$$

$$\implies \frac{d}{dt}(m_1\dot{\vec{r_1}} + m_2\dot{\vec{r_2}}) = 0 \tag{123}$$

$$\implies \frac{d}{dt}(M\dot{\vec{R}}) = 0 \tag{124}$$

$$M\dot{\vec{R}} = const \tag{125}$$

i.e. $\dot{\vec{R}}$ *moves with constant velocity in the absence of external forces* [8]. *Having established that the* S' *frame fixed to the CM is an inertial frame (thus equivalent to S) we may shift our primary reference frame to the CM frame, with the origin fixed to the CM of the binary system. As evident in figure 22, the position vector of the CM is now* $\vec{R} = 0$. *Further,* $\because$ $m_1\vec{r_1} + m_2\vec{r_2} = 0$ *in the CM frame and* $\vec{r} = \vec{r_1} - \vec{r_2}$,

[8] In the presence of external forces $M\ddot{\vec{R}}_{CM} = \vec{F}_{ext}$

$$\implies \vec{r_1} = \frac{m_2}{m_1+m_2}\vec{r} \qquad \vec{r_2} = \frac{-m_1}{m_1+m_2}\vec{r} \tag{126}$$

The L can be rewritten as;

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \tag{127}$$

with $\mu = \frac{m_1 m_2}{m_1+m_2}$, *the reduced mass of the binary system. For the Sun - Earth binary system the CM lies within the boundary of the Sun* $\because$ $M_{Sun} \ggg M_{Earth}$. *The CM of the Sun - Jupiter system lies outside the Sun.*[9] *Where does CM of the Earth - Moon system lie and is there any perceptible effect of that point that we experience on Earth?*

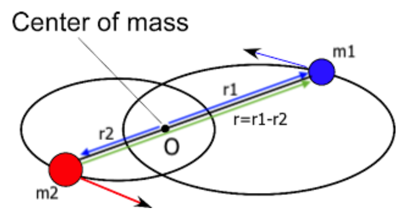


Figure 22: A binary system showing the centre of mass fixed at the common focii or the two elliptical orbits. $\vec{r} = \vec{r_1} - \vec{r_2}$

[9] Do you know what is the barycentre?

The form of the new Lagrangian shows that the two body problem is reduced to an equivalent one body problem, which is to determine the motion of a single particle of mass μ *in a central field, derivable from the potential* $V(r)$.

For the motion of a particle moving under a central force field (analysed from an inertial frame) space is isotropic. $\because$ *the mechanical properties of the system will be unaffected by the orientation of the system vis-a-vis its co-ordinate axes therefore the Lagrangian is also invariant under rotation. If a coordinate system is rotated about an arbitrary axis by* $d\theta$ *(figure 23), then the position vector of a point* $\vec{r}$ *changes to* $\vec{r} + d\vec{r}$ *given by* $d\vec{r} = d\vec{\theta} \times \vec{r}$. *Each and every vector also transforms the same way, e.g.* $d\dot{\vec{r}} = d\vec{\theta} \times \dot{\vec{r}}$

Say, the system has one particle and is rotated about some axis by $d\theta$,

$$dL = \sum \frac{\partial L}{\partial x_i} dx_i + \sum \frac{\partial L}{\partial \dot{x}_i} d\dot{x}_i = 0$$

will be invariant. Now $p_i = \frac{\partial L}{\partial \dot{x}_i}$ *and* $\dot{p}_i = \frac{\partial L}{\partial x_i}$

$$dL = \sum \dot{p}_i dx_i + \sum p_i d\dot{x}_i = 0$$

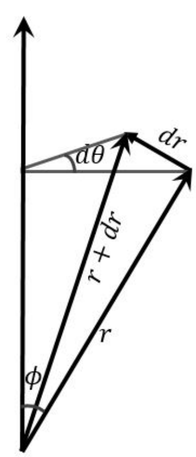


Figure 23: Vector transformation under rotation

Now, including all three co-ordinates.

$$\dot{\vec{p}}.d\vec{r} + \vec{p}.d\dot{\vec{r}} = 0$$
$$\dot{\vec{p}}.(d\vec{\theta} \times \vec{r}) + \vec{p}.(d\vec{\theta} \times \dot{\vec{r}}) = 0$$
$$d\vec{\theta}.(\vec{r} \times \dot{\vec{p}}) + d\vec{\theta}.(\dot{\vec{r}} \times \vec{p}) = 0$$
$$d\vec{\theta}.\frac{d}{dt}(\vec{r} \times \vec{p}) = 0$$
$$i.e.\ \frac{d}{dt}(\vec{r} \times \vec{p}) = 0 \implies \vec{r} \times \vec{p} = constant$$

$$\implies \vec{L} = \Sigma\, \vec{r}_i \times \vec{p}_i = constant \tag{128}$$

Thus, if space is isotropic the angular momentum ($\vec{L}$) is conserved and it is additive like $\vec{p}$. If external force field has an axis of symmetry then the Lagrangian of the system in that force field is invariant under rotation about that axis.

Going back to the 2 body problem. Since the conservative force field, corresponding to $V(r)$ is always directed along $\vec{r}$ and $\vec{L}$ = constant, implies that the initial value of L will be preserved throughout. Consequently, the plane containing the vectors $\vec{r}$ and $\vec{p}$ that defined $\vec{L}$ is also defined by that initial value of $\vec{L}$. The motion of the particle will thus be restricted to that plane (determined by the initial condition) and we can write the Lagrangian of the system as;

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\ (\theta \text{ is ignorable}) \tag{129}$$
$$\implies \dot{p_\theta} = \frac{\partial L}{\partial \theta} = 0 \tag{130}$$
$$\implies p_\theta = \mu r^2\dot{\theta} = const = l\ (say) \tag{131}$$

This is the EL EOM corresponding to the coordinate θ. In principle l can be positive or negative and has a simple geometric interpretation.

Remember Noether's Theorem: The connection between symmetry and invariance of associated physical quantities extend further;

- Homogeneity of space – conservation of linear momentum.
- Isotropy of space – angular momentum.
- Homogeneity of time – energy conservation

For a closed system, where the interaction forces are derivable from a potential, there are seven additive integrals of motion - energy E, linear momentum $\vec{p}$ and angular momentum $\vec{L}$; the latter two having 3 components each.

Kepler's Laws:

1. *The orbit of a planet is an ellipse with the Sun at one of the two foci.*
2. *A line joining planet to sun sweeps equal areas in equal intervals of time (figure 24).*

$$dA = \tfrac{1}{2}r^2 d\theta$$
$$\implies \frac{dA}{dt} = \tfrac{1}{2}r^2\dot{\theta}$$
$$= \tfrac{l}{2\mu} = const$$

dA/dt is the areal velocity. Note: This is a property of motion under a central force field and not unique to any particular $V(r)$.

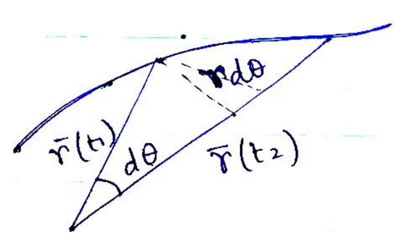


Figure 24: Areal velocity

3. *The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.* $T^2 \propto a^3$; T *=time period,* a *=length of semi major axis.*

Now, $\because$ *L is independent of time* ($\frac{\partial L}{\partial t} = 0$) *we have a conserved quantity namely the total energy* (E) *of the system.*

$$E = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{132}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + V(r) \tag{133}$$

$$\implies \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{\mu^2 r^2}\right)} \tag{134}$$

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{\mu^2 r^2}\right)}} \tag{135}$$

Equation 135, can be solved to get $r(t)$*. However, the equation of the trajectory may be readily obtained otherwise. Note,* $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$

$$\implies \theta(r) = \int \frac{l/r^2 dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}} \tag{136}$$

Thus the problem can be solved in principle in terms of **4** *constants:* r_0, θ_0, E *and* l.

You can't do this!

Now both the Lagrangian and Total Energy are dependent on $\dot{\theta}$,

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \qquad E = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{137}$$

it is tempting to replace $\dot{\theta}$ *with* $l/\mu r^2$*, in the equations and obtain,*

$$L = \frac{\mu\dot{r}^2}{2} + \frac{l^2}{2\mu r^2} - V(r) \qquad E = \frac{\mu\dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + V(r) \tag{138}$$

But, it is important to note that, while the replacement is justified in E we cannot replace $\dot{\theta}$ *in terms of r in L. Why? The Equation of Motion* $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0$ *are derived under the assumption that all coordinates and their variations are independent, but that breaks down with* $l = \mu r^2\dot{\theta}$.

Equation of motion and the orbit

The Euler Lagrange EOM for the coordinate r is derived as,

$$L = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{139}$$

$$\frac{\partial L}{\partial r} = \mu\dot{\theta}^2 r - \frac{\partial V}{\partial r}; \quad \frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \tag{140}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \mu\ddot{r} - \mu\dot{\theta}^2 r + \frac{\partial V}{\partial r} = 0 \tag{141}$$

$$\implies \mu\ddot{r} = -\frac{\partial V}{\partial r} + \frac{l^2}{\mu r^3} \tag{142}$$

The structure of the above equation i.e. $ma = F$ shows that the governing equation involves only 1 variable (r) thus the problem is reduced to effectively an 1D motion. Here, the force term (RHS of equation above) is derivable from a modified effective potential $U(r) = V(r) + \frac{l^2}{2\mu r^2}$, the effective force $\vec{F}_{eff} = -\nabla U(r)$ and the total energy of the system given by,

$$E = \frac{\mu\dot{r}^2}{2} + U(r) \tag{143}$$

A part of the kinetic term is now included in the potential component.

Problem 1

Our first problem deals with the inverse square law type force, $V(r) = -k/r$ and $F(r) = -\frac{k}{r^2}$.

$$V(r) = -\frac{k}{r} \tag{144}$$

$$U(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \tag{145}$$

at the minima of $U(r)$ i.e. $r = r_0$, $U'(r_0) = 0$

$$\frac{\partial V}{\partial r} - \frac{l^2}{\mu r^3} = 0 \tag{146}$$

$$\left.\frac{\partial V}{\partial r}\right|_{r_0} = \left.\frac{l^2}{\mu r^3}\right|_{r_0} \tag{147}$$

$$\left.\frac{\partial}{\partial r}(-k/r)\right|_{r_0} = \frac{k}{r_0^2} = \frac{l^2}{\mu r_0^3} \tag{148}$$

$$\implies r_0 = \frac{l^2}{\mu k} \tag{149}$$

$$U_{min} = U(r_o) = -\frac{k^2}{2\mu l^2}, \quad F(r_o) = -\frac{l^2}{\mu r_o^3} \tag{150}$$

If $E = E_1 \geq 0$, the motion is unbounded. However, there is a distance of closest approach r_{11} to the centre of force.

$U(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$
The second term of $U(r)$ is occasionally called the centrifugal potential
$F_c = -\frac{\partial}{\partial r}(\frac{l^2}{2\mu r^2}$
$F_c = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 = \mu\omega^2 r$

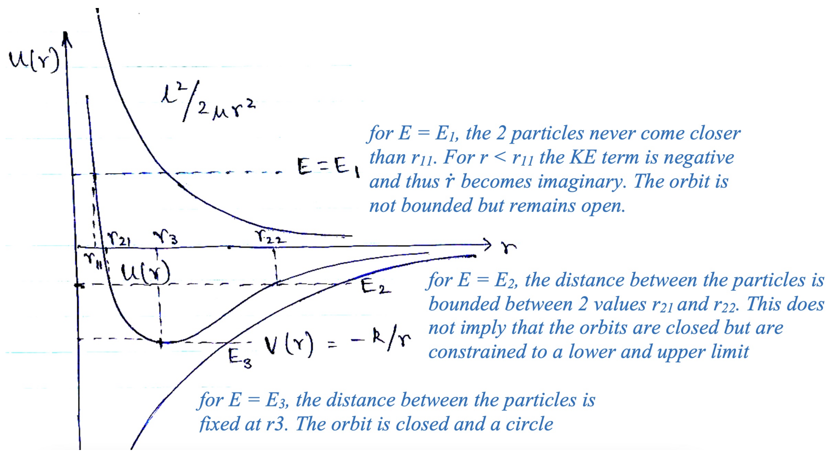


Figure 25: The 1D effective potential plot for inverse square force field for 2 interacting particles. Potentials corresponding to the inverse square force and the "centrifugal force" are also shown along with various values of total energy of the system and the resulting orbits allowed.

If $E = E_2 < 0$, bounded motion exists. The particle of effective mass μ moves between two limits of shortest (r_{21}) and longest distance (r_{22}) (see figure 25). Now visit the expression for $\dot{r}$,

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{\mu^2 r^2}\right)}$$

For $E = E_2$, $\dot{r} = 0$ at the turning points, thus (r_{21}) and (r_{22}) are given by the solutions to the equation [10]

$$E - V(r) - \frac{l^2}{\mu^2 r^2} = 0 \tag{151}$$

$$V = -\frac{k}{r} \implies -2Er - 2kr + \mu l^2 = 0 \tag{152}$$

$$\implies r_{11} \text{ and } r_{12} = \frac{k \pm \sqrt{k^2 + 2E\mu l^2}}{-2E} \tag{153}$$

[10] at the turning points the KE is NOT zero, only the radial component of velocity is zero.

To solve the equation of the trajectory we need some further juggling. We know that

$$L = \frac{1}{2}\mu(\dot{r^2} + r^2\dot{\theta}^2) - V(r) \tag{154}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{155}$$

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r} = F(r) \tag{156}$$

Let, $u = \frac{1}{r}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \tag{157}$$

$$\frac{du}{d\theta} = -\frac{\mu}{l}\dot{r} \tag{158}$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu}{l\dot{\theta}}\ddot{r} = -\frac{\mu^2}{l^2}r^2\ddot{r} \tag{159}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2u^2}F(r = 1/u) \tag{160}$$

This form is particularly useful if we want to find the force $F(r)$ *if the trajectory is already well known. For the inverse square law force* $F = -k/r^2 = -ku^2$ *and the above equation becomes;*

$$\frac{d^2u}{d\theta^2} + u = \frac{k\mu}{l^2} \tag{161}$$

which is a equation similar to that of a simple harmonic oscillator acted upon by a constant force and the "time" parameter replaced by θ*. That the coefficient of* u *on the LHS is 1 implies that the "frequency" of oscillation or repeat is* $\theta = 2\pi$ *and by consequence the orbit is closed.*[11] *Let's return to the equation of the trajectory,*

[11] Closure of the orbit is a special property of the inverse square force law. Additionally, $F = -kr$ is the other force law that supports closed orbits.

$$\theta(r) = \int_{r_0}^{r} \frac{\pm\frac{l}{r^2}dr}{\sqrt{2\mu\left(E - V - \frac{l^2}{2\mu r^2}\right)}} + \theta_0$$

Putting $u = \frac{1}{r}$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu V}{l^2} - u^2}} \tag{162}$$

Assume that $V = -kr^{n+1}$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu k u^{-(n+1)}}{l^2} - u^2}} \tag{163}$$

Terms in radical are of the form $\sqrt{A + Bx + Cx^2}$*. This can be integrated if* $-(n+1) = 0, 1, 2$*. Excluding* $n = -1$*, since that would imply* $V =$ *constant and interaction force is zero. For* $n = 1$ *the equation is also easily integrable.*

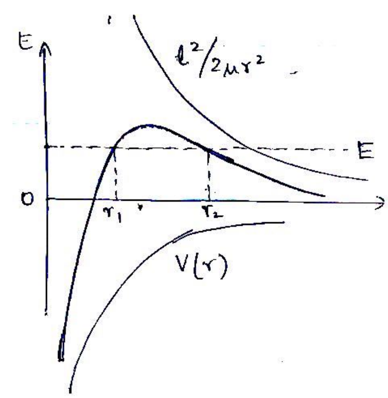


Figure 26: Effective potential for $V(r) = -k/r^3$

Problem 2: $V(r) = -\frac{k}{r^3}$

The figure above plots the effective potential for an inverse cube attractive potential illustrating the conditions for bounded, unbounded and circular *orbits. Note for $U_{max} > E > 0$, the distance from the force centre cannot exceed r_1 if $r \leq r_1$ and if $r \geq r_2$ then the particle cannot come closer than r_2. Thus the region between $r_1 < r < r_2$ is forbidden. A circular orbit though is possible for $E = U_{max}$.*

Problem 3: The Simple Harmonic Potential

$V(r) = -kr^2$, The effective potential is plotted in figure 27. For $l = 0$: The motion corresponds to a straight line trajectory passing through $r = 0$. For any $E > 0$. Motion is bounded by a $r_{max} = 2E/k$. For $l \neq 0$: Motion is bounded between a minimum r_1 and maximum r_2 values of r. At the Minima of $U(r)$ the trajectory is a circle.

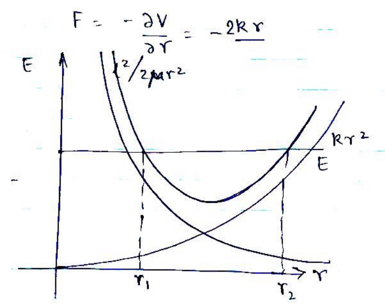


Figure 27: Effective potential

Planetary Motion - Kepler's Problem

Let's return to the case of $V = -k/r$, which corresponds to Newtonian Gravity.

$$\theta(r) = \int \frac{\frac{l}{r^2}dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} + const \tag{164}$$

which can be integrated with the substitution $u = 1/r$ and under the condition r = distance of closest approach (r_{min}) at $\theta = 0$.

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu k u}{l^2} - u^2}} \tag{165}$$

We make use of the result that;

$$\int \frac{du}{\sqrt{A + Bx + Cx^2}} = \frac{1}{\sqrt{-C}} \arccos\left(-\frac{B + 2Cx}{B^2 - 4AC}\right) \tag{166}$$

In the present case we have, $A = \frac{2\mu E}{l^2}$, $B = \frac{2\mu k}{l^2}$ and $C = -1$. Inserting the above terms and rearranging we get,

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos(\theta - \theta_0)\right) \tag{167}$$

assuming $\theta_0 = 0$ *we get,*

$$\cos\theta = \frac{l^2/\mu k r \quad -1}{\sqrt{1+\frac{2El^2}{\mu k^2}}} \tag{168}$$

$$\text{Say,} \qquad \alpha = \frac{l^2}{\mu k} \qquad \text{and} \qquad \epsilon = \sqrt{1+\frac{2El^2}{\mu k^2}} \tag{169}$$

$$\cos\theta = \frac{\alpha/r - 1}{\epsilon} \tag{170}$$

$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta \tag{171}$$

Equation 171 is the general equation of a conic section, ϵ = *eccentricity and* 2α = *latus rectum. The minimum value of r occurs when* $\theta = 0$ *or* $\cos\theta$ *is maximum. Changing the total energy of the particle changes the nature of the conic section by varying its eccentricity.*

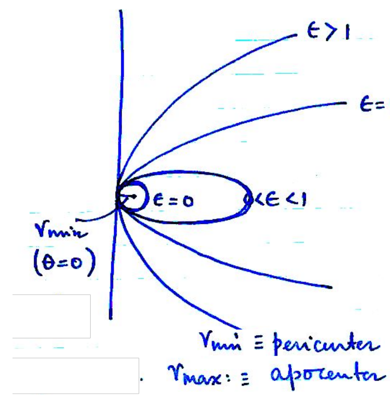


Figure 28: Trajectory for different ϵ

$$E > 0 \implies \epsilon > 1 \longrightarrow \text{Hyperbola} \tag{172}$$

$$E = 0 \implies \epsilon = 1 \longrightarrow \text{Parabola} \tag{173}$$

$$U_{min} < E < 0 \implies 0 < \epsilon < 1 \longrightarrow \text{Ellipse} \tag{174}$$

$$E = U_{min} \implies \epsilon = 0 \longrightarrow \text{Circle} \tag{175}$$

The solutions for parabola and the hyperbola correspond to unbounded trajectories as evident from the figure 25 for $E \geq 0$. *For bounded orbits as in planetary motion we know that the orbits are bounded and elliptical (*$E < 0$*) with major and minor axes given by,*

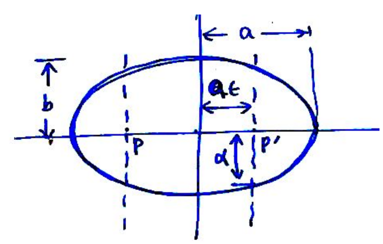


Figure 29: Elliptical Orbit

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|} \longrightarrow \text{decided by k and E} \tag{176}$$

$$b = \frac{\alpha}{\sqrt{1+\epsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \longrightarrow \text{decided by E and l} \tag{177}$$

$$\text{thus,} \qquad \frac{b^2}{\alpha} = a \tag{178}$$

$$r_{min} = a - a\epsilon = \frac{\alpha}{1+\epsilon} \tag{179}$$

$$r_{max} = a + a\epsilon = \frac{\alpha}{1-\epsilon} \tag{180}$$

Remember we had proved that $dA/dt = 2\mu/l$ that allows us to calculate the total Area swept in a complete period,

$$\int_0^\tau dt = \frac{2\mu}{l}\int_0^A dA \tag{181}$$

$$\tau = \frac{2\mu A}{l} \tag{182}$$

$$\tau = \frac{2\mu}{l}\pi ab \tag{183}$$

$$\tau = \frac{2\mu}{l}\pi\frac{k}{2|E|}\frac{1}{\sqrt{2\mu|E|}} \tag{184}$$

$$\tau^2 = \frac{\mu\pi^2k^2}{2l^2|E|^3} \tag{185}$$

For an ellipse we know that,

$$a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|}, \quad b = \frac{\alpha}{\sqrt{1-\epsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}, \implies \frac{b^2}{\alpha} = a \tag{186}$$

$$\implies \tau^2 = \left(\frac{2\mu\pi}{l}\right)^2 a^2b^2 = a^3\alpha\left(\frac{2\mu\pi}{l}\right)^2 \tag{187}$$

Kepler's third law: Ratio of square of τ to the cube of the semi major axis (a) is constant for all planets. Now, the proportionality constant $\tau^2/a^3 = 4\pi^2\mu/k$ was concluded by Kepler to be same for all planets, see figure 30. Now $k = Gm_1m_2$ and $\mu = \frac{m_{sun}m_2}{m_{sun}+m_2}$,

$$\frac{\tau^2}{a^3} = \frac{4\pi^2\mu}{k} = \frac{4\pi^2}{G(m_{sun}+m_2)} = C$$

Here, the constant C is dependent on the mass of both bodies. But, $\because$ the mass of the planets $m_2 << m_{sun}$

$$\frac{\tau^2}{a^3} = \frac{4\pi^2\mu}{k} \approx \frac{4\pi^2}{Gm_{sun}} = C$$

which is independent of the mass of the planet and C is empirically observed to be almost a constant.

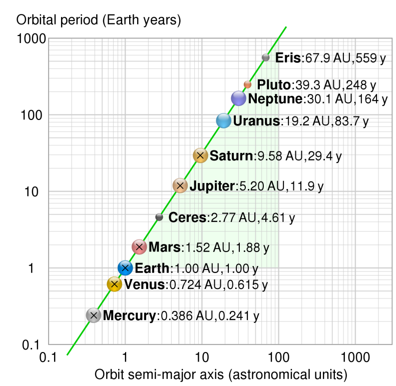


Figure 30: Log-log plot of τ vs a for various planets and other bodies of the solar system. (wikipedia)

The Laplace Runge-Lenz Vector

Kepler's problem is also distinguished by an additional conserved vector. Recall that the expressions for the angular momentum and Newton's second law are,

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times\mu\dot{\vec{r}} = constant$$

$$\dot{\vec{p}} = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}$$

The cross product between the 2 above vectors give,

$$\begin{aligned}
\dot{\vec{p}} \times \vec{L} &= \mu \frac{f(r)}{r}[\vec{r} \times \vec{r} \times \dot{\vec{r}}] \\
\dot{\vec{p}} \times \vec{L} &= \mu \frac{f(r)}{r}[\vec{r}(\vec{r}.\dot{\vec{r}}) - r^2\dot{\vec{r}}] \\
\because \vec{r}.\dot{\vec{r}} &= \frac{1}{2}\frac{d}{dt}(\vec{r}.\vec{r}) = r\dot{r} \ \ \textit{and} \ \ \dot{\vec{p}} \times \vec{L} = \frac{d}{dt}(\vec{p} \times L) \\
\implies \frac{d}{dt}(\vec{p} \times L) &= -\mu f(r) r^2 \left(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\dot{r}}{r^2}\right) \\
&= -\mu f(r) r^2 \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)
\end{aligned}$$

for $f(r) = \frac{-k}{r^2}$, *then*

$$\frac{d}{dt}(\vec{p} \times \vec{L}) = \frac{d}{dt}\left(\frac{\mu k \vec{r}}{r}\right) \tag{188}$$

$$\implies \frac{d}{dt}\left(\vec{p} \times \vec{L} - \frac{\mu k \vec{r}}{r}\right) = 0 \tag{189}$$

$$\implies \vec{A} = \vec{p} \times \vec{L} - \frac{\mu k \vec{r}}{r} = \textit{(constant)} \tag{190}$$

Vector $\vec{A}$ *is known as the Laplace Runge-Lenz Vector. We know that* $\vec{L}$ *is perpendicular to the plane of motion (figure 31), which gives,*

$$\vec{A}.\vec{L} = 0$$

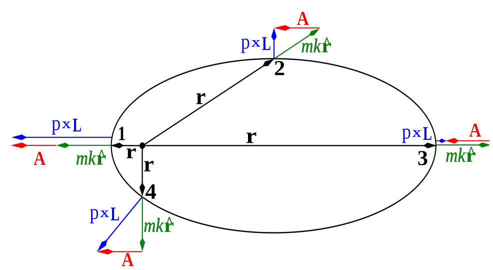


Figure 31: The Laplace-Runge-Lenz vector at different points on the orbit

It follows from the orthogonality of $\vec{A}$ *to* $\vec{L}$ *that* $\vec{A}$ *is a fixed vector in the plane of orbit.*

$$\begin{aligned}
\vec{A}.\vec{r} = Ar\cos\theta &= \vec{r}.(\vec{p} \times \vec{L}) - \mu k r \\
\vec{r}.(\vec{p} \times \vec{L}) &= \vec{L}.(\vec{r} \times \vec{p}) = l^2 \\
\vec{A}.\vec{r} = l^2 - \mu k r \qquad \textit{or} \qquad Ar\cos\theta &= l^2 - \mu k r \\
\textit{rearranging we get} \ \ \frac{1}{r} &= \frac{\mu k}{l^2}\left(1 + \frac{A}{\mu k}\cos\theta\right)
\end{aligned}$$

Remember, $\frac{\alpha}{r} = 1 + \epsilon\cos\theta$, *which gives us* $|\vec{A}| = \mu k \epsilon$ *i.e. A has the magnitude* $\mu k \epsilon$ *and the conservation of* $\vec{A}$ *is a second route of deriving the orbit equation as derived above.*

$$A = \mu k \epsilon \implies A^2 = \mu^2 k^2 + 2\mu E l^2 \tag{191}$$

A is a function of k, E *and* l.

Points to Note

- *Conserved quantities correspond to a symmetry of the problem. Which symmetry does the conservation of the* $\vec{A}$ *correspond to?*

- *The system was specified at* $t = 0$ *by 6 coordinates (3 position + 3 velocity) defining a 3 dimensional configuration space or a 6 dimensional phase space.*
- *A mechanical system with d degrees of freedom can have at most* $2d - 1$ *constants of motion, since there are* $2d$ *initial conditions and the initial time cannot be determined by a constant of motion.*
- *The seven scalar quantities derived by the 3 constants or conserved quantities* E, $\vec{A}$ *and* $\vec{L}$ *are related by two equations.* $\vec{A}.\vec{L} = 0$ *and* $A^2 = \mu^2 k^2 + 2\mu E l^2$, *giving five independent constants of motion. (Since the magnitude of* $\vec{A}$, *hence the eccentricity* ϵ *of the orbit, can be determined from the total angular momentum* $\vec{L}$ *and the energy* E, *only the direction of* $\vec{A}$ *is conserved independently; moreover, since* $\vec{A}$ *must be perpendicular to* $\vec{L}$, *it contributes only one additional conserved quantity.)*
- *The four constants* $\vec{L}$ *and* E *determine the shape of the orbit. The Laplace-Runge-Lenz vector adds one more constant that may be thought of as fixing the orientation of the orbit in the plane perpendicular to* $\vec{L}$.

Closed Orbits: Bertrand's Theorem

You can skip this section

We will side step the condition for closed orbits in general but consider the special case of Circular orbits that may exist at the minima of $U(r)$. At $r = r_o$, $\left.\frac{\partial V}{\partial r}\right|_{r_o} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2 = -F(r_o)$ and $\dot{r} = 0$. Thus the total energy of the system is given by,

$$E(r_o) = V(r_0) + \frac{l^2}{2\mu r_0^2} \tag{192}$$

Stability of Orbits

You can skip this section

In case the circular orbit results from a minima, then the orbit is stable, on the other hand, if it results from a maxima then the orbit is unstable. We know that,

$$U(r) = V(r) + \frac{l^2}{2\mu r^2} \tag{193}$$

The condition for stability arises from,

$$\frac{\partial^2 U}{\partial r^2} = \left.\frac{\partial^2 V}{\partial r^2}\right|_{r_0} + \frac{3l^2}{mr_0^4} \tag{194}$$

$$\frac{\partial^2 U}{\partial r^2} > 0 \quad \longrightarrow \quad Stable \tag{195}$$

$$\frac{\partial^2 U}{\partial r^2} < 0 \quad \longrightarrow \quad Unstable \tag{196}$$

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_0} = -\left.\frac{\partial F}{\partial r}\right|_{r_0} \tag{197}$$

$$Stability, \quad \Longrightarrow \quad -\left.\frac{\partial F}{\partial r}\right|_{r_0} > -\frac{3l^2}{mr_0^4} \tag{198}$$

$$But, \quad F(r_0) = \frac{-l^2}{\mu r_0^3} \tag{199}$$

$$\left.\frac{\partial F}{\partial r}\right|_{r_0} < -\frac{3F(r_0)}{r_0} \tag{200}$$

$$\left.\frac{\partial F/\partial r}{F/r}\right|_{r_0} < -3 \tag{201}$$

$$\left.\frac{d\, luF}{d\, lur}\right|_{r_0} < -3 \tag{202}$$

If $F = -\frac{k}{r^{n+1}} \Longrightarrow$ above condition given $n < 2$. Thus an inverse power law potential varying slower than $1/r^2$ is capable of circular orbits for all r_0

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2u^2}F(1/u) \tag{203}$$

For such circular orbits if E is sightly higher than U_{min} the particles motion is still bounded and given by,

$$u = u_0 + a\cos\beta\theta \tag{204}$$

Were, a =amplitude decide by ΔE.
$u = \frac{1}{r}$ $\longrightarrow$ *Dealt pertubatively retaining 1st terms of Taylor series exp.*

$$\beta^2 = 3 + \frac{r}{F}\left(\frac{dF}{dr}\right) \qquad or \qquad \beta^2 = 3 + \frac{dluF}{dlur} \tag{205}$$

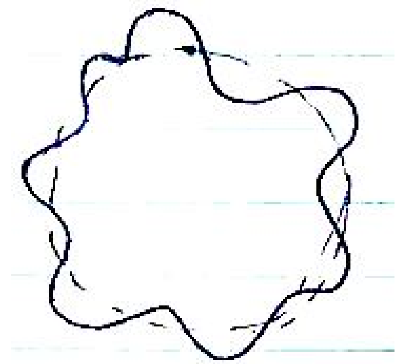


Figure 32: Perturbing a circular orbit. If β is rational then the orbit finally closes on itself.

Thus particle executes SHM about the stable orbit u_0.
The above condition leads to $F(r) = -\frac{k}{r^{3-\beta^2}}$

Thus all force laws of this form with β as a rational number lead to closed orbit if initial condition differ slightly from condition for circular orbit.

In case the deviation is more than slight such that it requires the higher order terms of Taylor series expansion. Now if we require that even for these longer deviation from circularity the orbit remain bounded.

It is found that for more than 1st order deviation from circlarity the orbit can still be closed if $\beta^2 = 1$ or $\beta^2 = 4$. i.e.

$$F(r) = -\frac{k}{r^2} \qquad or \qquad F(r) = -kr \tag{206}$$

Thus, $F(r)$ is either inverse square law or follows Hooks law. Conditions for

$$\text{Circular Orbit:} \qquad \left.\frac{\partial U}{\partial r}\right|_{r_0} = 0 \tag{207}$$

$$\text{Stability:} \qquad \left.\frac{\partial^2 U}{\partial r^2}\right|_{r_0} > 0 \tag{208}$$

$$\frac{\frac{\partial F}{\partial r}}{F/r} < -3 \tag{209}$$

Problem: *Investigate the stability of circular orbits in a force field $V(r) = -\frac{k}{r}e^{-\frac{r}{a}}$, for $k, a > 0$, called the screened Coulomb Potential. Now,*

$$U(r) = V(r) + \frac{l^2}{2\mu r^2} \tag{210}$$

$$\left.\frac{\partial U}{\partial r}\right|_{r_0} = \frac{k}{r_0^2}e^{-\frac{r_0}{a}} + \frac{k}{r_0 a}e^{-\frac{r_0}{a}} - \frac{l^2}{2\mu r^3} = 0 \tag{211}$$

$$\frac{ke^{-\frac{r_0}{a}}}{r_0}\,(\frac{1}{r_0} + \frac{1}{a}\bigg) = \frac{l^2}{2\mu r^2} \tag{212}$$

$$\frac{ke^{-\frac{r_0}{a}}}{r_0}\left(\frac{1}{r_0} + \frac{1}{a}\right) - \frac{l^2}{2\mu r^3} = 0 \tag{213}$$

From the condition for stability,

$$\left.\frac{\partial^2 U}{\partial r^2}\right|_{r_0} > 0 \quad (214)$$

$$-\frac{k}{r_0^2}e^{-\frac{r_0}{a}}\left(\frac{1}{r_0}+\frac{1}{a}\right)-\frac{ke^{-\frac{r_0}{a}}}{r_0 a}\left(\frac{1}{r_0}+\frac{1}{a}\right)-\frac{ke^{-\frac{r_0}{a}}}{r_0^3}+\frac{3l^2}{\mu r_0^4} > 0 \quad (215)$$

$$-\frac{ke^{-\frac{r_0}{a}}}{r_0}\left(\frac{1}{r_0^2}+\frac{1}{ar_0}+\frac{1}{r_0 a}+\frac{1}{a^2}+\frac{1}{r_0^2}\right)+\frac{3}{r_0}\frac{ke^{-\frac{r_0}{a}}}{r_0}\left(\frac{1}{r_0}+\frac{1}{a}\right) > 0 \quad (216)$$

$$-\frac{1}{r_0^2}-\frac{1}{r_0 a}+\frac{1}{a^2} < 0 \quad (217)$$

$$\frac{-a^2-r_0 a+r_0^2}{r_0^2 a^2} < 0 \quad (218)$$

$$r_0^2 - r_0 a - a^2 < 0 \qquad \textit{or} \qquad \left(\frac{a}{r_0}\right)^2+\frac{a}{r_0}-1 > 0 \quad (219)$$

$$\textit{Let, } z = \frac{a}{r_0} \qquad z^2+z-1 > 0 \quad (220)$$

For all values of z for which $z^2 + z - 1$ *equal to and larger than zero.*

$$z = \frac{1}{2}(\sqrt{5}-1) \approx 0.62 \quad (221)$$

So if $\frac{a}{r_0} \geq 0.62$ *and if E and l are such that it allows circular orbit.For the case* $a \longrightarrow \infty$, *then* $V(r) \longrightarrow -\frac{k}{r}$*: Stability is guaranteed.*

Scattering in a Central Force Field:

The theoretical analysis developed in the context of the Central Force motion dealt with the problem of planetary motion. But the analysis can be equally well be applied to scattering of particles interacting via central force fields. For example, scattering of charged particles by Coulomb interaction. Quantum effects are much stronger in such cases though many classical predications still remain valid and importantly the procedure for describing the scattering phenomena remain the same.

Scattering based on collisions happen either via contact interactions e.g. colliding billiard balls or via long range interactions e.g. scattering of α particles by the positively charged nucleus via the repulsive forces between them.

Scattering via a fixed Centre of Force

In the one body formulation the scattering problem is concerned with scattering of beam of particles (e.g. α particles) by a centre of force at the origin.[12]
Assumptions:

- $V(r) = k/r$ *(repulsive potential) and* $\vec{F} = k/r^2\hat{r}$
- $F(r) \longrightarrow 0$ *as* $r \longrightarrow \infty$
- *A beam of constant flux density (= Intensity (I); no. of particles/unit area) is incident from infinite distance to the force centre along a straight line.*

[12] The situation is akin to Rutherford scattering that deals with the elastic scattering of charged particles by the Coulomb interaction. The famous scattering experiment of scattering of alpha particles (He nuclei) with Au foil was done by Hans Geiger and Ernest Marsden in 1909, in collaboration with Rutherford. The observations were explained by Rutherford in 1911 that led to the development of the Rutherford model and eventually the Bohr model of the atom.

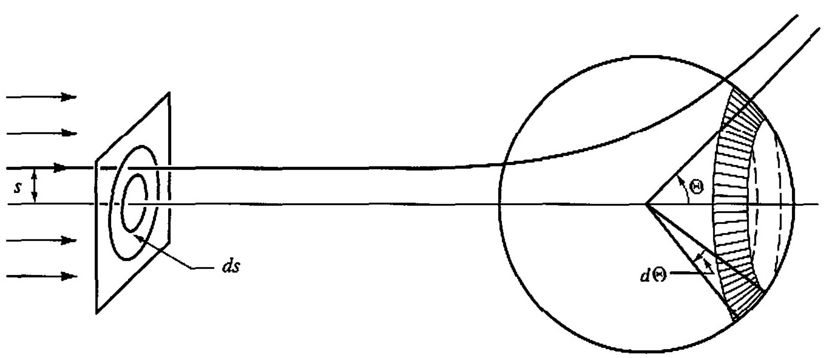


Figure 33: Trajectory of scattered particles

- *After Interaction (attractive/repulsive) the incident particle move away and asymptotically approach a straight line trajectory.*

The cross section for scattering in a given direction is given by $\sigma(\Omega)$

$$\sigma(\Omega)d\Omega = \frac{\textit{no. of particles scattered into solid angle } d\Omega}{\textit{incident intensity}} \tag{222}$$

$\sigma(\Omega)$ has the dimension of area and thus justifying its nomenclature as differential scattering Cross-Section. Now, the total solid angle subtended at the origin by a ring of scattered particles between angles Θ and $\Theta + d\Theta$ is $d\Omega = 2\pi \sin\Theta d\Theta$, and the degree of scattering of a particle is determined by the parameters l_o (initial angular momentum) and E_o (initial energy). Here, the initial parameters of any incident particle are the initial velocity (v_o) and the impact parameter (S), which defines the angular momentum and energy.

$$l_o = mv_oS \qquad \text{and} \qquad E = \frac{1}{2}mv_0^2 \tag{223}$$

$$\implies v_0 = \sqrt{\frac{2E}{m}} \text{ and } l_o = S\sqrt{2mE} \tag{224}$$

Thus, once E and S are determined, the trajectory of the particle is fixed along with the scattering angle.The number of particles scattered into the angle $\Theta \longrightarrow \Theta + d\Theta$ are the number of particles incident between $S \longrightarrow S + dS$, as shown in figure 33. Therefore per unit time,

$$2\pi S|dS| \times I = 2\pi \sin\Theta|d\Theta| \times \sigma(\Theta)I \tag{225}$$

$$\sigma(\Theta) = \frac{S}{sin\Theta}\left|\frac{dS}{d\Theta}\right| \tag{226}$$

The absolute signs are introduced since particles with smaller impact parameter are scattered by larger angles and vice versa. In reality the force centre is a fixed nucleus of charge $-Z_1e$ and the incident particle has charge $-Z_2e$ i.e. for a beam of α particles $Z_2 = 2$. The interaction force is then given by,

$$F = \frac{1}{4\pi\epsilon_o}\frac{Z_1Z_2e^2}{r^2} = \frac{k}{r^2}$$

with $k = \frac{Z_1Z_2e^2}{4\pi\epsilon_o}$. Energy of incident particles $E > 0$, the trajectories will be hyperbolic.

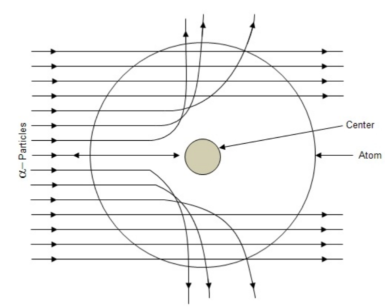


Figure 34: Particles with smaller impact parameter are scattered to larger angles and vice versa

$$\text{with} \qquad \epsilon = \sqrt{1 + \frac{2El^2}{mk^2}} = \sqrt{1 + \frac{2El^2}{m}\left(\frac{4\pi\epsilon_o}{Z_1Z_2e^2}\right)^2} \tag{227}$$

$$\because l = S\sqrt{2mE}, \;\; \epsilon = \sqrt{1 + \left(\frac{8\pi\epsilon_oES}{Z_1Z_2e^2}\right)^2} \tag{228}$$

As derived earlier, the equation of the trajectory is a conic section with the general equation $\frac{\alpha}{r} = 1 + \epsilon\cos\theta$, where $\alpha = \mu l/k$ and the angle θ is measured with respect to the axis defined by the line joining the point of closest approach, r_{min} to the origin i.e. the force centre, as shown by the red line in figure 35. The angle for the incoming and outgoing asymptotes (green lines in figure 35) Ψ are determined by $r \rightarrow \infty$ along the incoming and outgoing directions of the α particle. Thus for $r \rightarrow \infty$

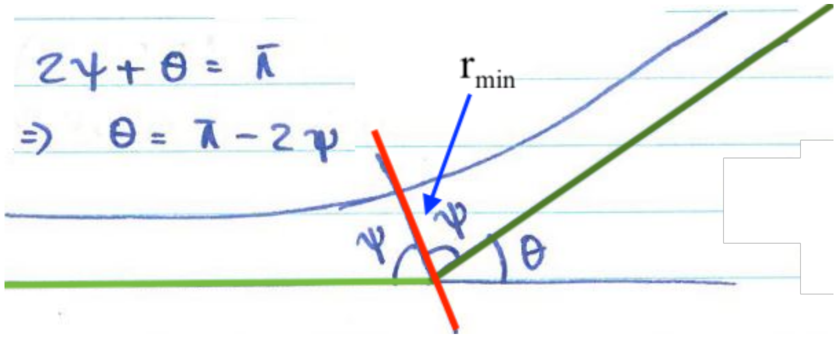


Figure 35: Scattering Angle (trajectory is symmetric about the red line)

$$\frac{\alpha}{r(\to\infty)} = 1 + \epsilon\cos\Theta(\to\Psi) \implies \cos\Psi = -\frac{1}{\epsilon} \tag{229}$$

$$\cos\frac{\pi-\Theta}{2} = \sin\frac{\Theta}{2} = -\frac{1}{\epsilon} \implies \cot^2\frac{\Theta}{2} = \epsilon^2 - 1 \tag{230}$$

$$\cot^2\frac{\Theta}{2} = \left(\frac{8\pi\epsilon_o ES}{Z_1 Z_2 e^2}\right)^2 \tag{231}$$

$$S = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_o E}\cot^2\frac{\Theta}{2} \tag{232}$$

$$\sigma(\Theta) = \frac{S}{\sin\Theta}\left|\frac{dS}{d\Theta}\right| = \frac{1}{4}\left(\frac{k}{2E}\right)^2\csc^4\frac{\Theta}{2} \tag{233}$$

$$\implies \sigma(\theta) \propto \csc^4\frac{\Theta}{2} \tag{234}$$

$$\textit{and} \quad \sigma(\Theta) \propto k^2 \tag{235}$$

Note that since $\sigma(\Theta) \propto k^2$, the above result is same for attractive or repulsive potential, even though the actual trajectories will be different. This is the main finding of the Rutherford's scattering formula albeit in the CM frame, where the problem is reduced to an one body problem. Quantum Mechanical calculations in the non-relativistic limits yields exactly the same result for scattering cross-section.

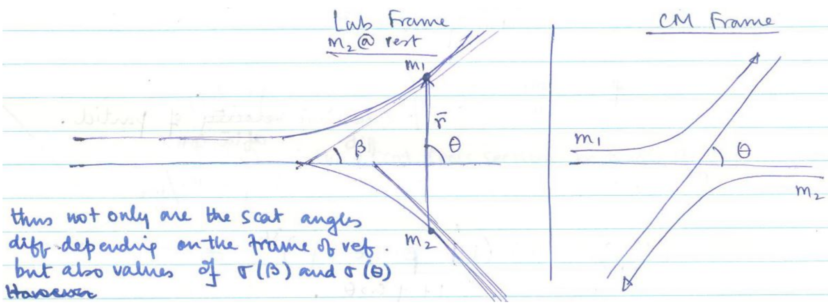
What is useful is the quantity σ_T, the integral of $\sigma(\Omega)$ in all directions, known as the total scattering cross section.

$$\sigma_T = \int_{4\pi}\sigma(\Omega)d\Omega = 2\pi\int_0^{\pi}\sigma(\Theta)\sin\Theta d\Theta \tag{236}$$

*and in the differential form, $\sigma(\Omega)$ gives the probability of scattering at a certain angle Θ. The σ_T for the coulomb field = ∞ (**Why?**) This is because the electrostatic force $(1/r^2)$ is infinite range and every trajectory, even for very large values of the impact parameter suffer some deflection. It can be shown that for force fields that decays faster than $1/r^2$ or are cut-off at finite distances results in a finite value of σ_T. For the coulomb field, such effects are provided by screening effects of other charges.*

Scattering in Laboratory Coordinates

Above calculation of $\sigma(\Theta)$ and the analysis was done with a fixed centre of force, which is akin to observing the phenomena from the CM frame as a one body problem. But how does the process appear in the lab frame where both the particles of masses m_1 and m_2 are allowed to move? We assume that initially m_2 was at rest and m_1 was incident with a velocity v_o.



The scattering angle of m_1, measured in the lab is β whereas we calculated σ in terms of Θ. Note: $\beta = \Theta$, if the m_2 is always at rest throughout the scattering process.
Say,

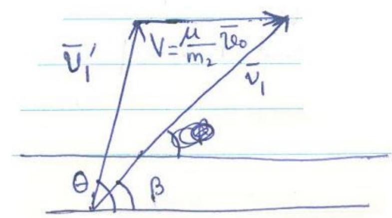
- *$\vec{r}_1$ and $\vec{v}_1$ = for particle m_1 after scattering. (Lab Frame)*
- *$\vec{r}_1'$ and $\vec{v}_1'$ = for m_1 after scattering. (CM Frame)*
- *$\vec{R}$ and $\vec{V}$ = for the CM of system. (Lab Frame)*
- *$\vec{V}$ is a constant with respect to the Lab Frame.*

Therefore at any instant we know,

$$\vec{r}_1 = \vec{R} + \vec{r}_1' \text{ and } \vec{v}_1 = \vec{V} + \vec{v}_1' \tag{237}$$

$$m_1\vec{v}_0 = (m_1 + m_2)\vec{V} \implies \vec{V} = \frac{\mu}{m_2}\vec{v}_0 \tag{238}$$

$$\vec{v}_1 = \frac{\mu}{m_2}\vec{v}_0 + \vec{v}_1' \tag{239}$$



$$\vec{v}_1' \sin\Theta = v_1 \sin\beta \tag{240}$$

$$v_1 \cos\beta = v_1' \cos\Theta + V \tag{241}$$

$$\text{taking the ratio of the above 2} \quad \tan\beta = \frac{\sin\Theta}{\cos\Theta + \rho}, \; \rho = \frac{\mu v_0}{m_2 v_1'} \tag{242}$$

$$\text{alternatively} \cos\beta = \frac{\cos\Theta + \rho}{\sqrt{1 + 2\rho\cos\Theta + \rho^2}} \tag{243}$$

$$\text{using, } v_1^2 = v_1'^2 + V^2 + 2Vv_1'^2\cos(\Theta) \tag{244}$$

If v is the relative speed of the particles after collision.

$$v_1' = \frac{m_2}{m_1 + m_2}v = \frac{\mu}{m_1}v$$

Note that the relative speed before collision was v_o and after collision is v. Together that gives us,

$$\rho = \frac{m_1}{m_2}\frac{v_0}{v} \tag{245}$$

Now if the collision is elastic $v_o = v$ and $\rho = m_1/m_2$. Though $\sigma(\beta)$ and $\sigma(\Theta)$ are different we note that the number of particles scattered into a given element of solid angle must be the same irrespective of the measurement angle.

$$2\pi T\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi T\sigma'(\beta)\sin\beta|d\beta| \tag{246}$$

$$\sigma'(\beta) = \sigma(\Theta)\frac{\sin\Theta}{\sin\beta}\left|\frac{d\Theta}{d\beta}\right| = \sigma(\Theta)\frac{d\cos\Theta}{d\cos\beta} \tag{247}$$

Here, $\sigma'(\beta)$ is the differential Scattering Cross-Section in terms of the angle β measured in the lab frame.

$$\sigma'(\beta) = \sigma(\Theta)\frac{(1+2\rho\cos\Theta+\rho^2)^{3/2}}{1+\rho\cos\Theta} \tag{248}$$

for elastic collision $v_0 = v$ and $\rho = \frac{m_1}{m_2}$, and in the case of equal masses $\rho = 1$

$$\cos\beta = \frac{\cos\Theta + 1}{\sqrt{2}\sqrt{1+\cos\Theta}} \tag{249}$$

$$= \sqrt{\frac{1+\cos\Theta}{2}} = \cos\frac{\Theta}{2} \tag{250}$$

$$\implies \beta = \frac{\Theta}{2} \tag{251}$$

For equal masses $\beta \leq \frac{\pi}{2}$ for $\Theta \leq \pi$

Homework

Central Force Motion

1. *What is the slope of the line in figure 30 and the intercept? Should those values be different for the various planets of the solar system?*

2. *Compute the ratio of the maximum and minimum speed of the Earth around the Sun given that the eccentricity of Earth's orbit about the Sun is 0.0167.*

3. *The equation of the path of a particle moving in a central force field with potential $V(r) = -k/r$ is given by*

$$\theta(r) = \int \frac{l/r^2 dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}}$$

Show that this can be integrated to give the equation of a conic section given by;

$$\cos\theta = \frac{l^2/\mu k r \quad - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}}$$

The origin and axes are so chosen that $\theta = 0$ at the point of closest approach.

4. *The Halley's comet is in an elliptical orbit around the Sun whose mass is 2×10^{30} kg. The eccentricity of the comet's orbit is 0.967, and the period is 76 years. Using these data, determine the closest and the farthest distance of Halley's comet from the Sun. Determine the speed of the comet when it is closest to the Sun?*

5. *For an artificial satellite in orbit around the Earth in an elliptical orbit the distances of closest and farthest approach are 10,000 km and 6,000 km, respectively. The mass of the satellite is 2000 kg. Calculate the eccentricity, energy, angular momentum, and minimum and maximum speeds of the satellite.*

6. *Two particles moving under the influence of their mutual gravitational attraction describe circular orbits about one another with a period τ. If they are suddenly stopped an allowed to gravitate towards each other, show that they will collide in time $t = \tau/4\sqrt{2}$.*

7. *Investigate the motion of a particle repelled by a force centre according to the law $F(r) = kr$, $k > 0$. Demonstrate that the orbit can only be hyperbolic.*

8. *A geostationary satellite is one whose position in the sky remains the same for an observer on the Earth. These satellites are placed above Earth's equator. Compute the radius of the orbit of a geostationary satellite.*

9. *A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an additional force $\vec{F} = -mC\vec{r}$, where m is the mass of the planet, C is a constant proportional to the gravitational constant and the density of the dust, and r is the radius vector from the Sun to the planet (both considered as points). This additional force is very small compared to the direct Sun-planet gravitational force. Plot the effective potential including the modified force field and the condition for a circular orbit. What would be the time period of the circular orbit?*

10. *A particle moves under a central force field $F(r) = -k/r^n$. Comment on the value of n if the resulting orbit is circular and passes through the centre of force.*

11. *A particle moves in an elliptical orbit in an attractive inverse square law central force field. If the ratio of the maximum angular velocity to minimum angular velocity is n, show that the eccentricity of the orbit is given by;*

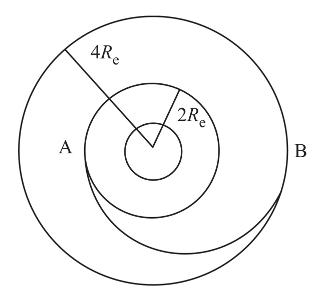
$$\epsilon = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

12. *Prove that among all the Kepler orbits of the same angular momentum, the circle has the least energy.*

13. *A particle moving in a central potential follows a trajectory given by $r\phi$ = const. Sketch the trajectory of the particle and compute the potential of the particle as a function of r.*

Advanced Problems

14. *A space vehicle of mass 2000 kg is orbiting around the Earth in a circular orbit with the radius of orbit as $2R_e$. We wish to transfer the vehicle to a circular orbit of radius $4R_e$. One of the schemes to transfer the vehicle is to use a semi- elliptical orbit as shown in the figure. What velocity changes are required at the points of intersection, A and B? What is the change in energy of the system in the two configurations?*



Trasferring between circular orbits

15. *Determine whether a particle moving on the inside surface of a cone under the influence of gravity can have a stable circular orbit.*

16. *Show that the motion of a particle in a central force field given by $F(r) = -k_1/r^2 - k_2/r^3$, where $(k_1, k_2 > 0)$ is described by a precessing ellipse.*

17. *What would be the shape of the orbit if the gravitational potential had a small correction that varies inversely with the square of distance? Which laws of planetary motion would still be valid? Goldstein Ch:3 problem 21*

18. *The Yukawa potential modifies the standard Coulomb potential by shortening its range and is given by* $V(r) = (V_0 r_0 / r) e^{-r/r_0}$ *. Find a particles trajectory in a bound orbit of the Yukawa potential to a first order in* r/r_0.

19. *The restricted three-body problem consists of two masses in circular orbits about each other and a third body of much smaller mass whose effect on the two larger bodies can be neglected. Define an effective potential* $V(x, y)$ *for this problem where the x axis is the line of the two larger masses. Sketch the function* $V(x, 0)$ *and show that there are two "valleys" (points of stable equilibrium) corresponding to the two masses. Also show that there are three "hills" (three points of unstable equilibrium). Goldstein Ch:3 problem 36*

Problems on Scattering

20. *From the* α *particle scattering problem how would you estimate the size of the fixed scatterer i.e. the size of the Au nucleus in the Rutherford scattering experiment?*

Small Oscillations

Oscillators appear almost everywhere in physics, starting from mechanics to string theory. It is one class of problems that can be solved exactly and leads to simple solutions that may be used to understand a wide variety of physical systems. Here we will look at oscillators with more than one degree of freedom. A simple harmonic oscillator with only one degree of freedom is familiar to everyone. We consider conservative systems in which the potential energy is function of only the generalised coordinates, $q_1, ..., q_n$ *and does not involve time explicitly, i.e. time dependent constraints are excluded. When the system is in equilibrium, the generalized forces vanish;*

$$Q_i = -(\frac{\partial V}{\partial q_i})_0 = 0 \tag{252}$$

The potential energy has an extremum at $q_{01}, q_{02}, ..., q_{0n}$*. If the system was initially at equilibrium, with zero initial velocities then it will continue to be in equilibrium. An equilibrium is stable if small disturbances do not lead to unbounded motion, it is unstable if small disturbances lead to unbounded motion and neutral if the equilibrium remains unchanged. Consider 3 spheres (A) a homogeneous sphere in which the centre of mass (CM) coincides with the geometric centre of the sphere (B) in which the CM lies vertically below the geometric centre and (C) the CM lies vertically above the geometric centre. These sphere rests on a surface with non-zero friction and are initially at equilibrium, neutral, stable and unstable, respectively. If each of the spheres are slightly displaced from their equilibrium they would behave very differently. (A) would not be perturbed at all (B) would return to its original position and continue to execute oscillatory motion and (C) would roll over! Their reactions are commensurate with the nature of their respective equilibria (A) neutral (B) stable (C) unstable.*

We are interested here in the motion of a system close to its stable equilibrium. Let the generalized coordinates deviate from the equilibrium positions by a small amount, η_i*,*

$$q_i = q_{0i} + \eta_i \tag{253}$$

Taylor expanding the potential about $q0i$ *we get*

$$V(q_i) = V(q_{i0}) + (\frac{\partial V}{\partial q_i})_0 \eta_i + (\frac{\partial^2 V}{\partial q_i \partial q_j})_0 \eta_i \eta_j + ... \tag{254}$$

where the summation convention has been used. The terms linear in η_i vanish due to the equilibrium condition. We can choose the zero of our energy scale at $V(q_{01}, q_{02}, ...q_{0n})$ so that we get

$$V(q_i) \simeq (\frac{\partial^2 V}{\partial q_i \partial q_j})_0 \eta_i \eta_j \tag{255}$$

by definition $(\frac{\partial^2 V}{\partial q_i \partial q_j})_0 = Vij = Vji$ (symmetric).

If the potential is independent of certain coordinates then the equilibrium is neutral along those coordinates. Any point along those directions is an equilibrium. The Vijs might also vanish at some points if V is linear in q_i. These cases need special treatment in the discussion below. Since the potential is conservative and independent of time, we know that the kinetic energy can also be written as a homogenous polynomial of order 2 in $\dot{q}_i$, then the Lagrangian becomes;

$$L = \frac{1}{2}(M_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j) \tag{256}$$

where The L then gives back the n EOMs.

M_{ij} may be functions of q_{i0}.

$$M_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0 \tag{257}$$

All n coordinates appear in each of the above n differential equations and these equations have to be solved simultaneously. In almost all cases of interest, the cross terms in the kinetic energy matrix are zero so that;

$$L = \frac{1}{2}(M_{ii}\dot{\eta}_i^2 - V_{ij}\eta_i\eta_j) \tag{258}$$

and the EOM corresponding to η_i are;

$$M_{ii}\ddot{\eta}_i + V_{ij}\eta_j = 0 \tag{259}$$

The equations of motion are linear differential equations with constant coefficients. We try an oscillatory solution:

$$\eta_i = Ca_i e^{-i\omega t} \tag{260}$$

The a_i are a complex amplitude and C is a scale factor introduced for convenience. Substituting in the EOM we get;

$$V_{ij}a_j - \omega^2 M_{ij}a_j = 0 \tag{261}$$

So we get n linear homogenous algebraic equations in a_i which have a solution if the determinant of the coefficient matrix is zero; i.e.

$$|\mathbf{V} - \omega^2\mathbf{M}| = 0 \tag{262}$$

Here V_{ij} and M_{ij} have been written as matrices **V** *and* **M**. *This determinant is a n^{th} order equation in ω^2 and the n solutions provide the right frequencies for which $Ca_ie^{-i\omega t}$ is the right solution. For these values of ω^2 we* *can find solution for the a_is.The equation also represents a type eigenvalue equation.*

$$\mathbf{Va} = \lambda \mathbf{Ma}, \qquad \lambda = \omega^2 \tag{263}$$

Importantly, we can show that;

1. *All λs and hence ω^2 are real*
2. *The eigenvectors* **a** *corresponding to different λ (ω^2) are real and are orthogonal in a sense*
3. *the matrix* **A** *made of all the vectors* **a** *will diagonalize both* **M** *and* **V**
4. **M** *will be diagonalised to the unit matrix i.e.* $\tilde{\mathbf{A}}\mathbf{M}\mathbf{A} = \mathbf{1}$
5. **V** *is diagonalised to a diagonal matrix whose elements are all the λs i.e.* $\tilde{\mathbf{A}}\mathbf{V}\mathbf{A} = \lambda$.

The matrix $\tilde{\mathbf{A}}$ is the transpose of the matrix **A**. *The transformation of the type $\tilde{\mathbf{A}}\mathbf{B}\mathbf{A}$ is called a congruence transformation and becomes a similarity transformation when* **A** *is orthogonal.*

Problem 1

Examine the two carts of figure 36, the EOMs of the 2 carts are given by;

$$m\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2 \tag{264}$$

$$m\ddot{x}_2 = k_2 x_1 - (k_2 + k_3)x_2 \tag{265}$$

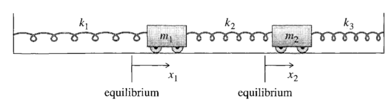


Figure 36: Two carts attached to fixed walls by the springs label k_1 and k_3, and to each other by k_2. The carts' positions x_1 and x_2 are measured from their respective equilibrium positions.

it is simplest to assume that when the two carts are at their equilibrium positions the three springs are neither stretched nor compressed i.e.their lengths are equal to their natural, unstretched lengths. However, depending on the distance between the two walls, it could be that all three springs are compressed or all three are stretched. Fortunately, as you can easily check, none of the results are affected by these possibilities.

Here we define 3 matrices

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{pmatrix}$$

Now the EOMs can be written in the beautifully compact matrix form;

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{V}\mathbf{x} \tag{266}$$

Suppose now that the two masses are equal, $m_1 = m_2 = m$, and similarly the three spring constants, $k_1 = k_2 = k_3 = k$. In this case, the matrices **M** *and* **V** *defined above reduce to;*

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 2k & -k \\ -k & 2k) \end{pmatrix}$$

The eigenvalue equation then becomes;

$$(\mathbf{V} - \omega^2 \mathbf{M})\mathbf{a} = \begin{pmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{pmatrix} \tag{267}$$

and its determinant must be set to zero to determine the non-trivial solutions;

$$|\mathbf{V} - \omega^2\mathbf{M}| = det \begin{pmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{pmatrix} = 0 \qquad (268)$$

which gives us two normal frequencies or the frequencies of the two normal modes that the dynamics of the system support;

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}} \qquad (269)$$

These two normal frequencies are the frequencies at which the system of carts can oscillate in purely sinusoidal motion. Now the actual motion is given by the column of real numbers $\mathbf{x}(t)$ *= Re* $z(t)$ *where the complex column* $\mathbf{z}(t) = C\mathbf{a}e^{i\omega t}$ *and* $\mathbf{a}$ *is made up of two real numbers,*

Notice that the first one, ω_1, is precisely the frequency of a single mass m on a single spring k.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

which must satisfy the eigenvalue equation

$$(\mathbf{V} - \omega^2\mathbf{M})\mathbf{a} = 0 \qquad (270)$$

Now that we know the possible normal frequencies, we must solve this equation for the vector $\mathbf{a}$ *for each normal frequency in turn. The sinusoidal motion corresponding to each of the normal frequencies are called the normal modes.*

For $\omega = \omega_1 = \sqrt{k/m}$*, the matrix* $(\mathbf{V} - \omega^2\,\mathbf{M})$ *becomes;*

$$(\mathbf{V} - \omega_1^2\mathbf{M}) = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

This matrix has determinant 0, as it should. Therefore, for this case, the eigenvalue equation reads;

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

which is equivalent to the two equations $a_l - a_2 = 0$ *and* $-a_1 + a_2 = 0$*. Both these equations are actually identical and either one implies that* $a_l = a_2 = e^{-i\delta}$*. The complex column z(t) is therefore given as;*

$$\mathbf{z}(t) = C \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega_1 t} = \begin{pmatrix} C \\ C \end{pmatrix} e^{i(\omega_1 t - \delta)}$$

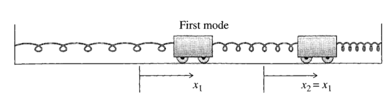


Figure 37: Mode 1: 2 carts oscillate sinusoidally, with equal amplitudes and in phase.

and the corresponding actual motion is given by the real column $\mathbf{x}(t) = Re\mathbf{z}(t)$ *or*

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} C \\ C \end{pmatrix} \cos(\omega_1 t - \delta)$$

We see that in the first normal mode the two carts oscillate **in phase** with each other and with the same amplitude C, as shown in figure 37. Since $x_i(t) = x_2(t)$, the middle spring is never stretched or compressed during the oscillations. For the first normal mode, the middle spring is actually irrelevant, and each cart oscillates just as if it were attached to a single spring and has a frequency $\omega_1 = \sqrt{k/m}$, same as for a single cart on a single spring.

For $\omega = \omega_2 = \sqrt{3k/m}$, the matrix ($\mathbf{V}$ - ω^2 $\mathbf{M}$) becomes;

$$(\mathbf{V} - \omega_2^2 \mathbf{M}) = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}$$

This matrix has determinant 0, as it should. Therefore, for this case, the eigenvalue equation reads;

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

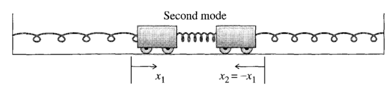


Figure 38: Mode 2: 2 carts oscillate sinusoidally, with equal amplitudes and out of phase.

which gives us $a_l + a_2 = 0$ or $a_1 = -a_2 = e^{-i\delta}$. The complex column z(t) is therefore given as;

$$\mathbf{z}(t) = C \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega_2 t} = \begin{pmatrix} C \\ -C \end{pmatrix} e^{i(\omega_2 t - \delta)}$$

and the corresponding actual motion is given by the real column $\mathbf{x}(t)$ = Re$\mathbf{z}(t)$ or

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} C \\ -C \end{pmatrix} \cos(\omega_2 t - \delta)$$

We see that in the second normal mode the two carts oscillate **out of phase** with each other again with the same amplitude C, as shown in figure 38. Since $x_i(t) = -x_2(t)$, when the first cart is displaced to the left the second cart will be displaced to the right and vice versa. Again when the outer springs are extended by a certain length the middle spring is compressed by twice the amount and vice versa. For the second normal mode, each cart is acted upon by force equivalent to 3 × a single spring thus each cart oscillates just as if it were attached to a single spring of force constant $3k$ and has a frequency $\omega_2 = \sqrt{3k/m}$.

Both the normal mode solutions;

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \delta_1) \qquad \mathbf{x}(t) = C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$$

satisfy the EOM $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{V}\mathbf{x}$ for any values of the four real constants $C_1, \delta_1, C_2, \delta_2$. Because the equation of motion is linear and homogeneous, the sum of these two solutions is also a solution;

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \delta_1) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2) \qquad (271)$$

This the general solution to the 2 second order differential equations with its four arbitrary constants. This general solution is rather difficult to visualize and describe. The motion of each cart is a mix of the two normal frequencies and the oscillatory pattern never repeats itself.

Remember the 2 original coupled EOMs

$$m\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m\ddot{x}_2 = k_2 x_1 - (k_2 + k_3)x_2$$

The coupling terms reflect the physical reality that the two carts are coupled to each other and that one cart cannot move without the other. It is possible to introduce alternative, so-called normal coordinates which, although less physically meaningful, they have the convenient property the EOMs decouple! This statement is true for any system of coupled oscillators, but is especially easy to see in the present case of two equal masses joined by three identical springs. In place of the coordinates x_1 and x_2, we can characterize the positions of the two carts by the two normal coordinates;

$$\xi_1 = \frac{1}{2}(x_1 + x_2) \tag{272}$$

$$\xi_2 = \frac{1}{2}(x_1 - x_2) \tag{273}$$

This is akin to a coordinate transformation given by;

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{274}$$

Show that in terms of the new coordinates ξ_1 and ξ_2 the EOMs in the present case of two equal masses joined by three identical springs transform to;

$$m\ddot{\xi}_1 = -k\xi_1 \tag{275}$$

$$m\ddot{\xi}_2 = -3k\xi_2 \tag{276}$$

which in the matrix notation reads;

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{pmatrix} = -\begin{pmatrix} k & 0 \\ 0 & 3k \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

The transformation from $(x_1, x_2) \rightarrow (\xi_1, \xi_2)$ seems to have diagonalised the matrix $\mathbf{V}$. Indeed if we look carefully at the transformation matrix $\mathbf{T}$ in equation 135 is composed of the 2 eigenvectors corresponding to the 2 eigen modes or normal modes of oscillation of the carts.

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \propto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

For the a_{ij}, the subscript i corresponds to the eigen (normal) mode index and j denotes the coordinate. Recall the statements;

1. *the matrix* $\mathbf{A}$ *made of all the eigenvectors* $\mathbf{a_i}$ *will diagonalize both* $\mathbf{M}$ *and* $\mathbf{V}$
2. $\mathbf{M}$ *will be diagonalised to the unit matrix i.e.* $\mathbf{\tilde{A}MA} = \mathbf{1}$
3. $\mathbf{V}$ *is diagonalised to a diagonal matrix whose elements are all the* λ*s i.e.* $\mathbf{\tilde{A}VA} = \lambda$.

Does the above transformation matrix satisfy the points 2 and 3 above? Lets see how we go about doing it. Recall the following that gives the general solution to the problem.

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \delta_1) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2) \tag{277}$$

And we demand that $\mathbf{\tilde{a}_i M a_i}$ *=1 and by consequence* $\mathbf{\tilde{A}MA} = \mathbf{1}$*, which is essentially a normalization. For the two eigenmodes we get;*

$$\mathbf{\tilde{a}_1 M a_1} = A_1^2 \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$\mathbf{\tilde{a}_2 M a_2} = A_2^2 \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$$

where A_1 *and* A_2 *may be interpreted as normalization constants. The above equations gives us* $A_1 = A_2 = 1/\sqrt{2m}$ *and the final form of the transformation matrix* $\mathbf{A}$ *as;*

$$\mathbf{A} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{278}$$

Thus the transformation;

$$\begin{aligned} \mathbf{\tilde{A}MA} &= \frac{1}{2m} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2m} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} m & m \\ m & -m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

What about the quantity $\mathbf{\tilde{A}VA}$*?*

$$\begin{aligned} \mathbf{\tilde{A}VA} &= \frac{1}{2m} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2m} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k & 3k \\ k & -3k \end{pmatrix} = \begin{pmatrix} k/m & 0 \\ 0 & 3k/m \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \end{aligned}$$

Thus the congruence transformation diagonalizes the $\mathbf{V}$ *matrix and yields the square of the eigenfrequencies of the problem. However, note that* $|\mathbf{A}| \neq \pm 1$.

Problem 2

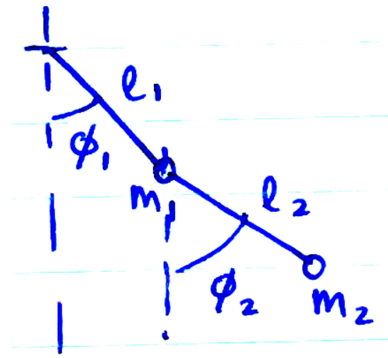


Figure 39: A double pendulum.

The Double Pendulum: Consider a double pendulum, comprising a mass m_1 suspended by a massless string of length l_1 from a fixed point, and a second mass m_2 suspended by a string of length l_2 from m_1, as shown in figure 39. The Lagrangian of the system is given by;

$$L = \frac{m_1+m_2}{2} l_1^2 \dot{\phi}_1^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{m_2}{2} l_2 \dot{\phi}_2^2$$
$$-[(m_1+m_2) g l_1 (1-\cos\phi_1) + m_2 g l_2 (1-\cos\phi_2)]$$

We can derive the EOMs via the Euler Lagrange equations but are too complicated to be solved analytically and gain any useful physical insight. So we make a couple of assumptions and approximations. Approximation 1: Just like the case of a simple pendulum we'll restrict our interest to the case of small oscillations i.e. the ϕ's are small. This allows us to rewrite L as;

$$L = \frac{m_1+m_2}{2} l_1^2 \dot{\phi}_1^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 + \frac{m_2}{2} l_2 \dot{\phi}_2^2$$
$$-[(m_1+m_2) g l_1 \phi_1^2 + m_2 g l_2 \phi_2^2]$$

where we have made the following approximations, $\cos(\phi_1-\phi_2) \simeq 1$ and $(1-\cos\phi_i) \simeq \phi_i^2$ Now the EOMs of the 2 generalized coordinates can be derived as;

Note: The exact expression the kinetic energy term was a transcendental function of the coordinates ϕ_i and velocities $\dot{\phi}_i$; the small-angle approximation reduced this to a homogeneous quadratic function of the two velocities only.

$$(m_1+m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 = -(m_1+m_2) g l_1 \phi_1 \tag{279}$$
$$m_2 l_1 l_2 \ddot{\phi}_1 + m_2 l_2^2 \ddot{\phi}_2 = -m_2 g l_2 \phi_2 \tag{280}$$

and the EOMs in the matrix format $\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{V}\boldsymbol{\phi}$ reads;

$$\begin{pmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = - \begin{pmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

To obtain the non-trivial solutions we have to demand that;

$$|\mathbf{V} - \omega^2 \mathbf{M}| = det \begin{pmatrix} (m_1+m_2)(g l_1 - \omega^2 l_1^2) & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 (g l_2 - l_2^2) \end{pmatrix} = 0$$

which yields a rather complicated quadriatic equation in ω^2. To simplify things further we now make an assumption that the masses and lengths are equal i.e. $m_1 = m_2 = m$, $l_1 = l_2 = l$ and define $\omega_0 = \sqrt{g/l}$. Now the above determinant reads;

$$|\mathbf{V} - \omega^2 \mathbf{M}| = det \begin{pmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & 2(\omega_0^2 - \omega^2) \end{pmatrix} = 0$$

which gives us the two eigenfrequencies

$$\omega_1^2 = (2 - \sqrt{2}) \omega_0^2 \tag{281}$$
$$\omega_2^2 = (2 + \sqrt{2}) \omega_0^2 \tag{282}$$

As before we now consider the eigenvalue equations $(\mathbf{V} - \omega^2 \mathbf{M})\mathbf{a} = 0$ *for the two cases to determine the eigenvectors.*

Case I: $\omega = \omega_1$

$$(\mathbf{V} - \omega_1^2 \mathbf{M})\mathbf{a} = ml^2\omega_0^2(\sqrt{2}-1)\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$\implies a_2 = \sqrt{2}a_1$ *i.e. the two bobs oscillate in-phase with the amplitude of the second pendulum* $\sqrt{2}\times$ *that of the first.*

Case II: $\omega = \omega_2$

$$(\mathbf{V} - \omega_2^2 \mathbf{M})\mathbf{a} = ml^2\omega_0^2(\sqrt{2}+1)\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$\implies a_2 = -\sqrt{2}a_1$ *i.e. the two bobs oscillate out of phase, with the amplitude of the second pendulum* $\sqrt{2}\times$ *that of the first. And the general solution to the motion of this special double pendulum is;*

$$\boldsymbol{\phi}(t) = C_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\cos(\omega_1 t - \delta_1) + C_2\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\cos(\omega_2 t - \delta_2) \qquad (283)$$

Again demanding that;

$$\tilde{\mathbf{a}}_1\mathbf{M}\mathbf{a}_1 = A_1^2 ml^2\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = 1$$

and

$$\tilde{\mathbf{a}}_2\mathbf{M}\mathbf{a}_2 = A_2^2 ml^2\begin{pmatrix} 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} = 1$$

we get the values $A_1^2 = 1/(2ml^2(2+\sqrt{2}))$ *and* $A_2^2 = 1/(2ml^2(2-\sqrt{2}))$. *Determine the transformation matrix* $\mathbf{A}$ *and show that the congruence transformation the* $\mathbf{M}$ *matrix to a unity matrix.*

Homework

1. *A hard rubber cylinder of radius r is held fixed with its axis horizontal, and a wooden cube of side* $2b$ *is balanced on top of the cylinder, with its center vertically above the cylinder's axis and four of its sides parallel to the axis (figure 40) The cube cannot slip on the rubber of the cylinder, but it can of course rock from side to side. By examining the cube's potential energy, find out if the equilibrium with the cube centered above the cylinder is stable or unstable. If stable find the frequency of small oscillations of the cube.*

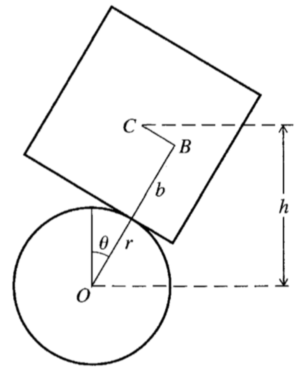


Figure 40: A cube, of side 2b and center C, is placed on a fixed horizontal cylinder of radius r and center o. It is originally put so that C is centered above o, but it can roll from side to side without slipping.

2. *Consider a simple harmonic oscillator with period r. Let* $< f >$ *denote the average value of any variable* $f(t)$*, averaged over one complete cycle:*

$$< f >= \frac{1}{\tau}\int_0^{\tau} f(t)dt \tag{284}$$

Prove that <T> = <U> = E/2 where E is the total energy of the oscillator. [Hint: Start by proving the more general, and extremely useful, results that $< \sin(\omega t - \delta) >=< \cos(\omega t - \delta) >= 1/2$*. Explain why these two results are almost obvious, then prove them by using trig identities to rewrite* $\sin^2\theta$ *and* $\cos^2\theta$ *in terms of* $\cos(2\theta)$*.*

3. *A spring of force constant k and zero unextended length is anchored at a point at one end and attached to a point mass m at the other end, as shown in the figures 41 (a) and (b). In (a) the point mass is constrained to move along a horizontal line at a perpendicular distance l from the anchoring point and in (b) the point mass is constrained to move along the arc of a circle of radius a. The distance between the anchor point and the centre of circle being* $l + a$*. Show that the frequency of small oscillation of the masses in the 2 cases are given by (a)* $\omega = \sqrt{k/m}$ *and (b)* $\omega = \sqrt{k(a+l)/ma}$*.*

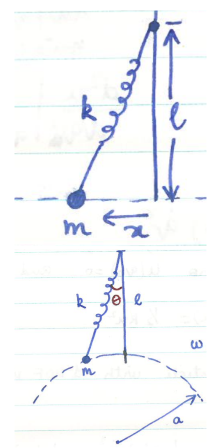


Figure 41: (a) and (b)

4. *The potential energy of a one-dimensional system with a point mass m at a distance r from the origin is,*

$$U(r) = U_0\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right) \tag{285}$$

for $0 < r < \infty$ *, with* U_0*, R, and* λ *all positive constants. Find the equilibrium position* r_0*. Let x be the distance from equilibrium and show that, for small x, the PE has the form* $U = const + kx^2$*. What is the angular frequency of small oscillations?*

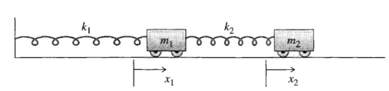


Figure 42: 2 carts and 2 springs

5. *(a) Find the normal frequencies for the two carts shown in figure 42 Assuming that* $m_1 = m_2$ *and* $k_1 = k_2$*. (b) Find and describe the motion for each of the normal modes in turn.*

6. *Two masses m_1 and m_2 are connected by a spring. When m_1 is held fixed, m_2 is observed to oscillate with a frequency ω. Find the frequency of linear oscillations when m_2 is held fixed and when both masses are free to move. You can neglect the effect of gravity.*

7. *Find the normal frequencies of a linear triatomic molecule which is modelled by 2 equal masses (m) connected to a central mass (M) by two identical springs of force constant k (figure 43). Show that the eigenfrequencies are given by $\omega_1 = 0, \omega_2 = \sqrt{\frac{k}{m}}, \omega_3 = \sqrt{\frac{k}{m}(1+\frac{2m}{M})}$. Analyse the motion and discuss the physical nature of the three eigenmodes.*

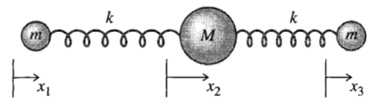


Figure 43: Triatomic molecule

8. *A pendulum consists of a mass M at the end of a massless string of length D. It is free to swing in one direction only, so has one degree of freedom, θ. The frequency of small oscillations is $\omega_0 = \sqrt{g/D}$, with g being the acceleration of gravity. Now suppose the string is very slowly shortened by some external agent. The length D varies only a little during the oscillation period. You can assume that $\theta(t) = A\cos\omega t$. where A , ω vary slowly with time but can be treated as constants over times of the order of a single period of the swinging pendulum. $D(t)$is a given function of the time and is not a dynamical variable. Treating θ as small;*
(a) Find the kinetic and potential energies. Find the Lagrangian and prove that the equation of motion is

$$\ddot{\theta} + \frac{2\dot{D}}{D}\dot{\theta} + \omega_0^2\theta = 0$$

(b) Notice that the total energy of the pendulum E = T + V is no longer constant in time because there is a term proportional to $\dot{\theta}$ in the EOM. The total energy also does not equal $H = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L$. Explain why this fact could be deduced from the form of T. Show that $E = H + m\dot{D}^2$.
(c) Find $\frac{dE}{dt}$ from $\frac{dH}{dt}$. Hint: Use the fact that L contains the time explicitly.
(d) The energy stored in the pendulum oscillations $E_{pend} = \frac{m}{2}(D^2\dot{\theta}^2 + Dg\theta^2) \simeq \frac{m}{2}DgA^2$, where A is the amplitude of oscillations and $\dot{\theta} \simeq -\omega A\sin\omega t$. $E > E_{pend}$ because, even if the pendulum doesn't swing, kinetic and potential energy both change. Part of your formula for $\frac{dE}{dt}$ vanishes if A = 0. This must be $\frac{dE_{pend}}{dt}$. Find it,and average it over one oscillation to get $\overline{\frac{dE_{pend}}{dt}}$. Here the "bar" means an average over one complete period. Prove that

$$\frac{1}{E_{pend}}\overline{\frac{dE_{pend}}{dt}} = -\frac{1}{2}\frac{\dot{D}}{D}$$

Use this formula to show that, for small oscillations, the energy stored in the oscillations of the pendulum increases as $\frac{1}{\sqrt{D}}$, no matter how we change D as long as we do it sufficiently slowly.

e) Show that the amplitude A of oscillations is proportional to $D^{\frac{3}{4}}$. The importance of this result is that $\frac{E_{pend}}{\omega}$ is a constant for any arbitrary variation of the pendulum length $D(t)$, as long as it varies slowly enough so that we can assume A, ω are constant for at least one period. Such a quantity, which remains constant for sufficiently slow variations in a parameter of the problem, is called an adiabatic invariant. This result played an important role in the early history of quantum theory. What you have shown applies to any system near its stable equilibrium point and not only to a pendulum.

Hamiltonian Formalism Canonical Transformations & Poisson Brackets

The Lagrangian and the Hamiltonian formalisms give two distinct yet interjected parts of analytical mechanics. The Hamiltonian formalism provides an elegant geometric picture of dynamical systems, and importantly serves as the starting point for quantum mechanics. The Hamiltonian formulation introduces a new variable, the canonically conjugate momentum $p = \frac{\partial L}{\partial \dot{q}}$. p though conjugate to q is also dynamically independent from q unlike the pair q and $\dot{q}$, used in the Lagrangian formulation which are not functionally independent but related by the time derivative i.e. $\dot{q} = \frac{dq}{dt}$. Use of q and p also leads to equations of motion that are symmetric about the two variables. Earlier we have introduced the configuration space in which we used the concept of variation to identify the extremal path that minimized the action and yielded the correct progression of $q(t)$ in time. Here on it will be convenient to use the more general $2n$ dimensional phase space, composed of all the coordinates $q(t)$ and the momenta $p(t)$, which are defined for the general case of n DoF by;

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad i = 1, 2, 3....n. \tag{286}$$

by the definition p is a function of q and $\dot{q}$, which are themselves functions of the time via the equations of motion. The concept of canonical momentum is the key concept in Hamilton's theory and loses its familiar definition $\vec{p} = m\vec{v}$. Which may be incidentally true in many simple cases, but no longer true in general, especially in cases involving generalized coordinates that are not necessarily position coordinates.

William Rowan Hamilton (1805-1865) was the astronomer royal for Eire. He worked extensively on planetary motion and on ray optics, for designing telescopes. Amazingly he found great similarity in the mathematical formulation of the laws of mechanics and ray optics. Indeed he wondered whether the laws of mechanics were the short wavelength approximation of some more general mechanical principles. see wikipedia

The transform is named after the French mathematician Adrien-Marie Legendre (1752–1833), who is also noted for establishing the modern notation for partial derivatives, which was subsequently adopted by Jacobi.

In thermodynamics we can transform between the Helmholtz and Gibbs free energies by using Legendre transforms. The Helmholtz free energy is often the most useful thermodynamic potential when temperature and volume are held constant, while the Gibbs energy is often the most useful when temperature and pressure are held constant. Similarly, internal energy $(U(S, V, N))$ and enthalpy $(H(S, P, N))$ can be transformed from one to the other. The latter gaining significance in problems where pressure is held constant.
see Legendre Transforms for Dummies and Legendre Transforms

Obtaining H from L

In physics, the Legendre transform commonly appears in thermodynamics and classical mechanics. It is used to convert functions of one quantity (such *as position, pressure, or temperature) into functions of the conjugate quantity (momentum, volume, and entropy, respectively). In classical mechanics we use Legendre transform to derive the Hamiltonian formalism out of the Lagrangian formalism and in thermodynamics to derive thermodynamic potentials.*

The Legendre transformation is a recipe for starting with a function of multiple variables that may be grouped into two sets active and passive and generating a new function where the passive variables remain but the active variables are replaced by a new set of variables. If the transformation is repeated, it restores the old function of the old variables.

In case of the Lagrangian to Hamiltonian transformation given by[13];

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L(\dot{q}_i, q_i, t) \tag{287}$$

[13] we have already encountered the Hamiltonian H in EL equation of the 2nd kind in equation 69

here, q_i are the passive variables and $\dot{q}_i$ are the active variables that get replaced by p_i. But, how does that happen?

remember this form of the Euler Lagrange equation is valid for systems with holonomic constraints and forces derivable from a potential

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{288}$$

$$\implies \dot{p}_i = \frac{\partial L}{\partial q_i} \tag{289}$$

Thus $p_i = f(\dot{q}_i, q_i, t)$ in principle may be inverted to give $\dot{q}_i = F(p_i, q_i, t)$ thus allow us to eliminate $\dot{q}_i$ from the RHS of equation 287 making H a function of p, q, t only.

Hamilton's equations of motion

Since the Legendre transformation can be made equally well in either direction, why do we prefer the variable p and the Hamiltonian $H(q, p)$ to the choice of $\dot{q}$ and $L(\dot{q}, q, t)$? The key feature of using the canonical momentum p, which is the tangent to the Lagrangian, instead of $\dot{q}$, is that Hamilton's Principle holds for dynamically independent variations of q and p. The arbitrary variations[14] *δp and δq are truly independent at each point in time, unlike the variations δq and $\delta \dot{q}$, which were never independent. Recall the discussion on principle of least action applied on the Lagrangian*[15];

[14] in the 2nD phase space

[15] the subscripts i are dropped for simplicity

$$\delta S = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \delta(p\dot{q} - H) dt = 0 \tag{290}$$

$$\delta L = \dot{q}\delta p + p\delta\dot{q} - \delta H \tag{291}$$

$$\delta H = \frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial q}\delta q \tag{292}$$

$$\delta L = (\dot{q} - \frac{\partial H}{\partial p})\delta p - (\dot{p} + \frac{\partial H}{\partial q})\delta q + \frac{d}{dt}(p\delta q) \tag{293}$$

The requirement that $\int \delta L dt = 0$ then indicates that the last term on RHS of equation 293 is zero $\because$ δq vanishes at the end points. The validity *of the Hamilton's principle requires that action (S) will be extremal for independent arbitrary variations of p and q if and only if the coefficients of δp and δq are identically zero. That gives us;*

$$\dot{q} = \frac{\partial H}{\partial p} \quad \& \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{294}$$

Now lets look at the differential of the function $H(p, q, t)$.

$$dH = \dot{q}dp - \dot{p}dq + \frac{\partial H}{\partial t}dt \tag{295}$$

$$\implies \frac{dH}{dt} = \dot{q}\dot{p} - \dot{p}\dot{q} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{296}$$

where we have used the equations 154 to replace the coefficients of the first 2 terms on the RHS. Now equation 287 also tells us that;

$$H = p\dot{q} - L(\dot{q}, q, t) \tag{297}$$

$$\implies dH = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt \tag{298}$$

$$\implies dH = \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt \tag{299}$$

$$\implies dH = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t}dt \tag{300}$$

$$\implies \frac{dH}{dt} = \dot{q}\dot{p} - \dot{p}\dot{q} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} \tag{301}$$

Along with the equations in 156 a comparison of equations 158 and 163 then gives us the Hamilton's equations of motion;

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \& \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t}} \tag{302}$$

We have reinserted the subscripts corresponding to the generalized coordinates of the problem.

In contrast to the Lagrangian formalism that gives n 2nd order differential EOMs the Hamiltonian formalism yields $2n$ first-order EOMs. The fact that q and p are treated symmetrically allows for the discovery of some important theorems i.e. Liouville's Theorem, the Poincare Recurrence Theorem.

Problem 1

The simple harmonic oscillator in Hamiltonian formalism. Consider the single particle system with the Lagrangian;

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{303}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{304}$$

$$\implies H = p\dot{x} - L \tag{305}$$

$$= m\dot{x}^2 - \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} \tag{306}$$

$$H(p, x) = \frac{p^2}{2m} + \frac{kx^2}{2} \tag{307}$$

and the Hamilton's EOM for $\dot{p}$ and $\dot{q}$ are;

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \& \quad \dot{p} = -\frac{\partial H}{\partial x} = -kx \tag{308}$$

Note that since the Lagrangian is not explicitly a function of time $\implies \frac{dH}{dt}$ = 0 i.e. H is a constant of motion and is also the total energy of the system. The simplest way to solve these two equations is to eliminate p between the equations above and we obtain our familiar second order differential equation $m\ddot{x} = -kx$ with the equally familiar solution[16]

$$x = A\cos(\omega t - \delta) \qquad p = \frac{\partial L}{\partial \dot{x}} = -m\omega A \sin(\omega t - \delta) \tag{309}$$

[16] Note: we can also eliminate x between the 2 first order differential equations and obtain a differential equation in p $\ddot{p} = -\frac{k}{m}p$

The phase space for the 1D oscillator is 2D with coordinates (x, p) where the above solutions for x and p is the parametric equation of an ellipse. The way the above solutions are written implies that x is maximum ($x = A$) at $t = 0$ and as t increases x decreases and p increases, thus the ellipse is traced in the clockwise direction. Figure 44, shows two orbits for the cases that the oscillator started out from rest at $x = A$ (solid curve) and $x = A/2$ (dashed curve). That the orbits have to be ellipses follows from conservation of energy i.e. H = total energy = $\frac{1}{2}m\omega^2A^2$ which allows us to write the equation of the ellipse in the phase space as;

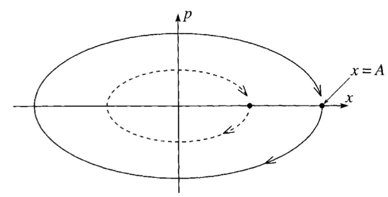


Figure 44: Elliptical trajectory for the 1D simple harmonic oscillator in phase space

$$\frac{x^2}{A^2} + \frac{p^2}{(m\omega A)^2} = 1$$

Thus the Hamiltonian formalism along with the phase space plots does give us some extra insight into the problem than before. Unfortunately, none of these examples exhibit any of the significant advantages of the Hamiltonian over the Lagrangian approach; rather, the Hamiltonian approach is just an alternative (at times circuitous) route to the same final EOMs.

Problem 2: Atwood's Machine

Problem 3: Particle in a Central Force Field

Using the plane polar coordinates (r, θ), the Hamiltonian can be written as,

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r) \tag{310}$$

Since the coordinate θ is a cyclic coordinate, i.e. it does not appear explicitly in the Hamiltonian, its conjugate momentum p_θ is a constant of the motion, of magnitude l of the angular momentum. The Hamilton's equations of motion for the θ coordinate are given by

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} = \frac{l}{mr^2} \tag{311}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = 0 \implies p_\theta = constant \tag{312}$$

Similarly, the Hamilton's EOM for r is

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{313}$$

$$\dot{p_r} = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{dV}{dr} \tag{314}$$

Taking the second derivative of r with respect to time and using Hamilton's equation of motion for p_r yields the radial EOM;

$$\ddot{r} = \frac{1}{m}\frac{dp_r}{dt} = -\frac{1}{m}\left(\frac{\partial H}{\partial r}\right) = \frac{p_\varphi^2}{m^2r^3} - \frac{1}{m}\frac{dV}{dr} = \frac{L^2}{m^2r^3} + \frac{1}{m}F(r) \tag{315}$$

Problem 4

Hamilton's Equations for a point mass on a cone

The Relativistic Lagrangian

The selling point of the Hamiltonian formalism i.e. writing the Hamiltonian for a system followed by obtaining the Hamilton's EOM is the physical identification of the total energy of a system with the Hamiltonian. As an example consider a relativistic free particle, with the corresponding Hamiltonian given by;

This is extra material that may be skipped

$$H = E = \sqrt{m_o^2c^4 + p^2c^2} \tag{316}$$

where m_o is the rest mass of the particle, c is the velocity of light and p the momentum. We can obtain the Lagrangian of the system as,

$$\begin{aligned} H &= \sum p_i\dot{q}_i - L \\ \implies H &= \sum p_i\frac{\partial H}{\partial p_i} - L \\ \implies L &= \sum p_i\frac{\partial H}{\partial p_i} - H \\ &= \sum p_i\frac{p_ic^2}{H} - H \\ &= (p^2c^2 - H^2)/H \\ &= \frac{-m_o^2c^4}{\sqrt{m_o^2c^4 + p^2c^2}} \\ &= m_oc^2\sqrt{1 - v^2/c^2} \end{aligned}$$

And the action integral for the relativistic free particle can be written as,

$$S = \int m_oc^2\sqrt{1 - v^2/c^2}dt \tag{317}$$

using the variational principle $\delta S = 0$ then yields the correct equations of motion. Note that the Lagrangian is not the KE of the particle.

Canonical Transformations

You will recognise that using the right coordinate system e.g. exploiting a inherent symmetry of a system eases the path towards finding the most conducive form of the EOMs and likely the ensuing solution. For example for a single particle in a central force field it is immensely helpful to use the spherical polar system than the Cartesian coordinates. Such transformations between coordinates is known as point transformations. In the phase space, however, the most general transformation would involve both the p's and q's and are known as contact transformations. Canonical transformation are a sub-class of contact transformations.

To begin with recall Hamilton's principle which says;

$$\delta \int L\, dt = 0 \quad \longrightarrow \quad \delta \int (p_i \dot{q}_i - H)\, dt = 0.$$

to a path in config space.

valid in a path in phase space → mod H principle (2n dim)

applies to H principle as well.

Hamilton's Eqns followed with the stipulation that $\delta q_i(1) = \delta q_i(2) = 0$. without requiring that $\delta p_i(1)$ or $\delta p_i(2) = 0$

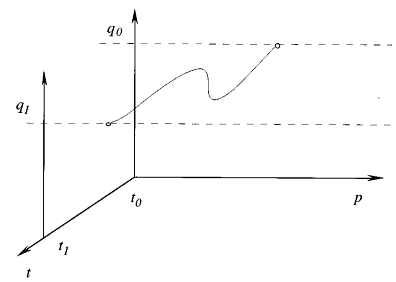


Figure 45: Phase space action including t as an axis. δq = 0 but not necessarily δp = 0, at the endpoints.

The end conditions are $q(t_0) = q_0$ and $q(t_1) = q_1$. However, the boundary condition on p is natural in the sense that there are no stipulations at the end points. We have a sort of "clothesline" boundary condition as depicted in the figure, where the trajectory is free to slide along the lines of constant q along the p direction. adapted from Hamiltonian description of the ideal fluid. P. J. Morrison, Rev. Mod. Phys. 70, 467, 1998

Thus we can define a phase space action as,

$$S(q,p) = \int (p_i \dot{q}_i - H) dt \tag{318}$$

the variation of which is zero for the correct path in phase space. Also recall that the L is not unique[17] *but for any arbitrary function $F(q,t)$ we can create a new Lagrangian $L' = L + \frac{d}{dt}F(q,t)$ that will yield the same EOMsThe same is true in the phase space, we can always migrate from L to L' as;*

$$L = p_i \dot{q}_i - H \rightarrow L' = p_i \dot{q}_i - H - \frac{d}{dt}F(p_i, q_i, t) \tag{319}$$

the difference being that in the phase space $F(q_i, p_i, t)$ may be a function of the p_i's also. But we still demand that δF vanishes at the end points which now requires both δq and δp = 0 at the end points. This non-uniqueness of the L and H has important consequences that may be effectively exploited.

[17] see discussion on pages 37 and 38

Now, consider a coordinate transformation of the $2n$ *variables* $(q_i, p_i) \rightarrow (Q_i, P_i)$ *from one phase space to another;*

$$Q_i = Q_i(q_j, p_j, t) P_i = P_i(q_j, p_j, t) \tag{320}$$

obviously many such transformations are possible but we are interested in those that satisfy the conditions below, involving a function $K(P_i, Q_i, t)$*, which is defined in the new transformed phase space and is equivalent to the original Hamiltonian.*

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \& \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{321}$$

demanding that;

- *there exists* K *in* P, Q *space such that* $\delta \int (P\dot{Q} - K)dt = 0$
- *the transformed coordinates* P, Q *are canonical coordinates*

Such transformations are known as canonical transformations. The validity of Hamilton's principle in both the phase spaces essentially requires that,

$$\delta \int (p\dot{q} - H)dt = \delta \int (P\dot{Q} - K)dt = 0 \tag{322}$$

This does not *imply that the integrands are equal but that they are related as;*

This would also imply that; $dF = -pdq + PdQ + (H - K)dt$ is an exact differential. A condition for a transformation to be canonical.

$$p_i\dot{q}_i - H - \frac{dF}{dt} = P_i\dot{Q}_i - K \tag{323}$$

By definition, Canonical transformations take us from one set of coordinates q *and their conjugate momenta* p *to another set* (P, Q) *in a way that the structure of the Hamilton's equation are preserved. The motivation of these transformations is to make as many of the coordinates (or momenta) cyclic or ignorable as possible in the transformed equivalent Hamiltonian, thus maximising the number of constants of motion. For each cyclic coordinate we will have a conserved momentum and so on. Indeed if all the coordinates and momenta can be made cyclic then the resulting Hamiltonian will be a constant (why not 0?) and will trivially yield all the constants of motion in terms of the initial conditions, without having to solve the EOMs! This is a unique way of exploiting an associated symmetry of the problem to yield a conserved quantity.*

The function F *above in equation 323 in general is a hybrid function i.e.* $F = F(q, p, Q, P, t)$ *and defines the transformation, hence is known as the generating function. Indeed, a* F *is useful for specifying the exact form of the canonical transformation only when half of the variables (beside the time) are from the old set and half are from the new. It then acts, as it were, as a bridge between the two sets of canonical variables and is called the generating function of the transformation*[18]*. To demonstrate the nature and flexibility afforded by a generating function, consider the case where*

[18] see Goldstein Chapter 9

the function $F = F(q, Q, t)$. The above equality (equation 323) would then require;

$$P_i\dot{Q}_i - K = p_i\dot{q}_i - H - \frac{dF}{dt} \quad (324)$$

$$= p_i\dot{q}_i - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \quad (325)$$

If either side of the above equation is to be exactly equal then;

$$K = H + \frac{\partial F}{\partial t} \quad (326)$$

$$p_i = \frac{\partial F}{\partial q_i} \quad (327)$$

$$P_i = -\frac{\partial F}{\partial Q_i} \quad (328)$$

Therefore given F we can obtain the new set of canonical coordinates that satisfy Hamilton's principle in the new phase space. Such transformations are known as canonical transformation. However all canonical transformations will not be derivable from a function of the form $F(q, Q, t)$ but other sets of old and new coordinates. As an example consider the following;

$$\delta \int (p_i\dot{q}_i - H - \frac{dF}{dt})dt = 0 \quad (329)$$

$$\delta \int (\frac{d}{dt}(p_i q_i) - \dot{p}_i q_i - H - \frac{dF}{dt})dt = 0 \quad (330)$$

$$\delta \int (-\dot{p}_i q_i - H - \frac{d}{dt}(F - p_i q_i))dt = 0 \quad (331)$$

$$\delta \int (-\dot{p}_i q_i - H - \frac{dF_3}{dt})dt = 0 \quad (332)$$

Here we have $F_3 = F - pq$ as our new generator function. Now we choose that $F_3 = F_3(p_i, Q_i, t)$ and therefore the integrand of the last equation (332) becomes

$$-\dot{p}_i q_i - H - \frac{dF_3}{dt} = -\dot{p}_i q_i - H - \frac{\partial F_3}{\partial p_i}\dot{p}_i - \frac{\partial F_3}{\partial Q_i}\dot{Q}_i - \frac{\partial F_3}{\partial t} \quad (333)$$

Equality of the above equation (332) with $\delta \int (P\dot{Q} - K)dt = 0$ then requires that;

$$K = H + \frac{\partial F_3}{\partial t} \quad (334)$$

$$q_i = -\frac{\partial F_3}{\partial p_i} \quad (335)$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \quad (336)$$

Note that self consistency ensuring the validity of the canonical nature of the *transformations for the above 2 cases would also demand;*

$$\frac{\partial p_i}{\partial Q_j} = \frac{\partial^2 F}{\partial Q_j \partial q_i} = -\frac{\partial P_j}{\partial q_i} \neq 0 \qquad (337)$$

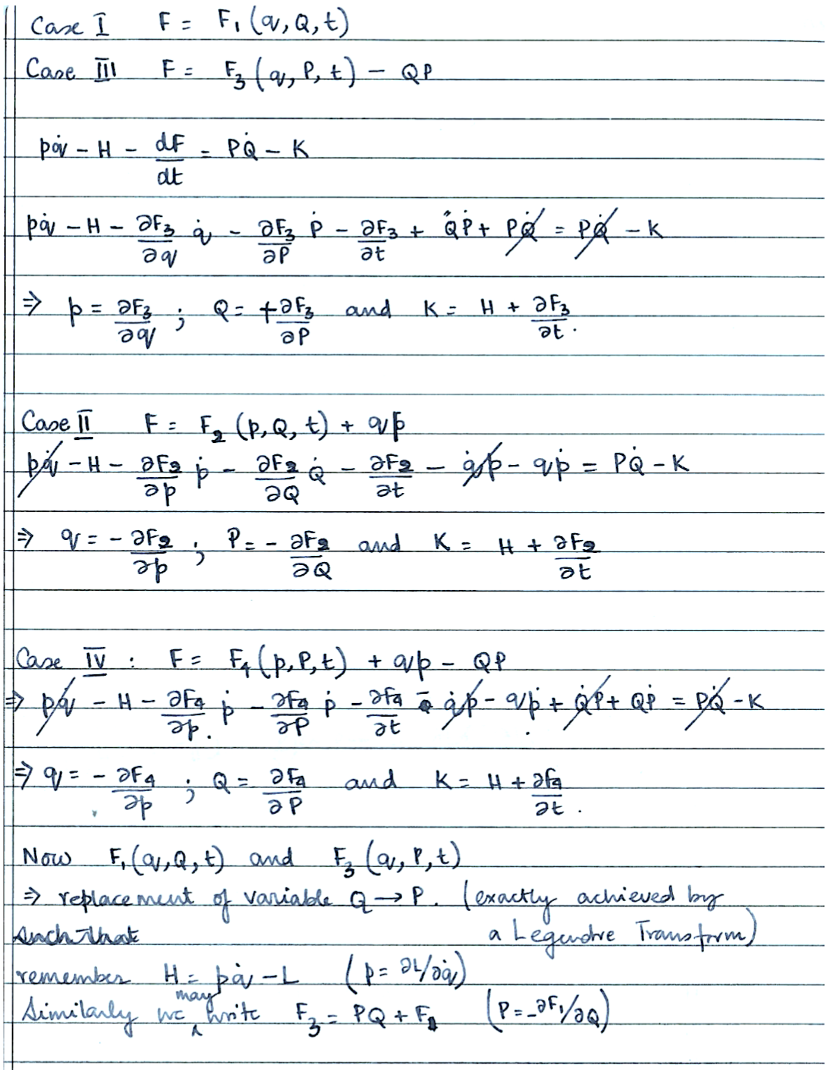
$$\frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i} \qquad (338)$$

*Table **??** below (adapted from Goldstein) summarises the 4 basic types of generating functions and their derivatives that also yield the actual canonical transformations, along with some special trivial cases. Note that each type involves half the variables of the original coordinates and half the new coordinates. See figure 46 for further details.*

Generating Function	Derivatives	Special Case
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$ $P_i = \frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i, Q_i = p_i, P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$ $Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i, Q_i = q_i, P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i, Q_i = -q_i, P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i, Q_i = p_i, P_i = -q_i$

It is important to note that these 4 functions are by no means complete, but show the most useful basic forms of the generating functions. The transformation equations (derivatives of the generating function) are obtained by matching the coefficients of the differentials on either side of the equation 324. Note that the 4 basic forms of the generating functions may be found by repeated use of the Legendre transformations replacing the dependence from one variable to the next. To summarise the following steps need to be executed to effect a canonical transformation.

- *Define F*
- *Use the corresponding derivatives of F i.e. relations in the middle column of table **??** to set up the transformation equation*
- *Use the relation* $K = H + \frac{\partial F}{\partial t}$ *to find the transformed Hamiltonian*



Case I $F = F_1(q, Q, t)$

Case III $F = F_3(q, P, t) - QP$

$p\dot{q} - H - \frac{dF}{dt} = P\dot{Q} - K$

$p\dot{q} - H - \frac{\partial F_3}{\partial q}\dot{q} - \frac{\partial F_3}{\partial P}\dot{P} - \frac{\partial F_3}{\partial t} + \dot{Q}P + P\dot{Q} = P\dot{Q} - K$

$\Rightarrow p = \frac{\partial F_3}{\partial q}$; $Q = +\frac{\partial F_3}{\partial P}$ and $K = H + \frac{\partial F_3}{\partial t}$.

Case II $F = F_2(p, Q, t) + qp$

$p\dot{q} - H - \frac{\partial F_2}{\partial p}\dot{p} - \frac{\partial F_2}{\partial Q}\dot{Q} - \frac{\partial F_2}{\partial t} - \dot{q}p - q\dot{p} = P\dot{Q} - K$

$\Rightarrow q = -\frac{\partial F_2}{\partial p}$; $P = -\frac{\partial F_2}{\partial Q}$ and $K = H + \frac{\partial F_2}{\partial t}$

Case IV : $F = F_4(p, P, t) + qp - QP$

$\Rightarrow p\dot{q} - H - \frac{\partial F_4}{\partial p}\dot{p} - \frac{\partial F_4}{\partial P}\dot{P} - \frac{\partial F_4}{\partial t} - \dot{q}p - q\dot{p} + \dot{Q}P + Q\dot{P} = P\dot{Q} - K$

$\Rightarrow q = -\frac{\partial F_4}{\partial p}$; $Q = \frac{\partial F_4}{\partial P}$ and $K = H + \frac{\partial F_4}{\partial t}$.

Now $F_1(q, Q, t)$ and $F_3(q, P, t)$

$\Rightarrow$ replacement of variable $Q \to P$. (exactly achieved by a Legendre Transform)

~~such that~~

remember $H = p\dot{q} - L$ $(p = \partial L / \partial \dot{q})$

Similarly we may write $F_3 = PQ + F_1$ $(P = -\partial F_1 / \partial Q)$

Figure 46: Calculating the transformations from the 4 basic generating functions.
Important: Case II and III are interchanged between data presented in figures 46 and the table above.

Problem 5

Consider the generating function $F_1(q, Q, t) = qQ$

$$\implies p = \frac{\partial F_1}{\partial q} \quad \& \quad P = -\frac{\partial F_1}{\partial Q} \tag{339}$$

which gives us the transformation $p = Q$ *and* $P = -q$*, effectively changing the role of the momenta and coordinates, demonstrating that the physical demarcation between generalised momenta and coordinates, is some sense, becomes irrelevant in canonical transformations.*

Problem 6

Consider the generating function $F_3(p, Q, t) = -pQ$

$$\implies q = -\frac{\partial F_3}{\partial p} \quad \& \quad P = -\frac{\partial F_3}{\partial Q} \tag{340}$$

which gives us the trivial identity transformation $q = Q$ *and* $p = P$*. Indeed the canonical transformations form a group, which necessitates the existence of the identity transformation and that the combination of any 2 canonical transformation is also a canonical transformation.*

Problem 7

Consider the generating function $F_3(p, Q) = -pQ + \epsilon H$*, where* ϵ *is a parameter independent of* q, p, Q, P*.*

$$\implies q = -\frac{\partial F_3}{\partial p} = Q - \epsilon \frac{\partial H}{\partial p} = Q - \epsilon \dot{q} \tag{341}$$

$$\implies P = -\frac{\partial F_3}{\partial Q} = p + \epsilon \frac{\partial H}{\partial Q} = p + \epsilon \dot{P} \tag{342}$$

Now, say $\epsilon = dt$

$$\implies Q = q + \epsilon \dot{q} \tag{343}$$

$$P = p + \epsilon \dot{p} \tag{344}$$

Note: we have replaced $\dot{P}$ *by* $\dot{p}$ *in the above equation and this is acceptable because as demonstrated through the equations the difference between them will be infinitesimal. Together the last 2 equations tell us that the canonical transformation gives us the values of the variables* q *and* p *at a translated time by* $+dt$*. The generating function importantly involves the Hamiltonian of the system. Again the inverse canonical transformation would have translated time backwards by* $-dt$ *and repeated applications of the same would have given us the variables at their initial values or the values given at any specified instant. This is akin to saying that all variables have been made cyclic.*

Problem 8

Consider the SHO whose $H = \frac{p^2}{2m} + \frac{m^2\omega^2 q^2}{2m}$. The mathematical form of the Hamiltonian tells us that since it is a sum of 2 squares a transformation of the form $p = \sqrt{f(P)}\cos Q$ and $q = \sqrt{f(P)}/m\omega \cos Q$ would make the coordinate Q cyclic, where $f(P)$ is some function of P.

$$H' = \frac{f(P)}{2m}\cos^2 Q + \frac{m^2\omega^2}{2m}\frac{f(P)}{m^2\omega^2}\sin^2 Q \tag{345}$$

$$= \frac{f(P)}{2m} \tag{346}$$

The modified H' should also satisfy Hamilton's EOMs i.e.

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{1}{2m}\frac{\partial}{\partial P}f(P) \quad \& \quad \dot{P} = -\frac{\partial H'}{\partial Q} = -\frac{1}{2m}\frac{\partial}{\partial Q}f(P) = 0 \tag{347}$$

Since $f(P)$ is quite arbitrary choose it such that $\dot{Q} = \omega$, which is a constant. Thus $Q(t) = \omega t + \alpha$ and $P(t) = A(constant)$.

$$\implies f(P) = 2mP\omega \tag{348}$$

$$\implies p = \sqrt{2mA\omega}\cos(\omega t + \alpha) \tag{349}$$

$$\implies q = \sqrt{2A/m\omega}\sin(\omega t + \alpha) \tag{350}$$

The constant A can be very easily identified in terms of the total energy E of the system as $A = E/\omega$.The p, q phase space plot of the SHO is obviously an ellipse as discussed earlier. What about the trajectory of the SHO in the P, Q phase space? Obviously, $P = E/\omega$ at all times i.e. for all values of Q which implies that the plot is a straight line in the P, Q space, that lies parallel to the Q axis and intersects the P axis at E/ω.

Poisson Brackets

Poisson brackets are algebraic constructs that are of greater significance to Quantum Mechanics. At the outset lets get familiar with the rules of these algebraic constructs. Let α and β be any two functions of q_i and p_i, and we define the Poisson bracket of these two functions as;

$$[\alpha, \beta] = \sum \frac{\partial \alpha}{\partial q_i}\frac{\partial \beta}{\partial p_i} - \sum \frac{\partial \alpha}{\partial p_i}\frac{\partial \beta}{\partial q_i} \tag{351}$$

which immediately implies the following identities;

1. $[\beta, \alpha] = -[\alpha, \beta]$
2. $[\alpha, \alpha] = 0$
3. $[c\alpha, d\beta] = cd[\alpha, \beta]$
4. $[\alpha, \beta + \gamma] = [\alpha, \beta] + [\alpha, \gamma]$

5. $[\alpha + \gamma, \beta] = [\alpha, \beta] + [\gamma, \beta]$

6. $[q_i, q_k] = 0; \quad [p_i, p_k] = 0; \quad [q_i, p_k] = \delta_{ik}$

7. $[\alpha, [\gamma, \beta]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0$ *(Jacobi identity)*

One of the most useful demonstrations of the Poisson bracket is by showing that $[L_x, L_y]$=L_z. **L = r**× **p** *and it is left as homework to prove that* $[L_x, L_y]$=L_z*, which is of significance in quantum mechanics.*

Another interesting application is to demonstrate temporal evolution of any variable $\alpha(p, q, t)$ *i.e. the EOMs of* α*. We can easily show that;*

$$\frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial q}\dot{q} + \frac{\partial \alpha}{\partial p}\dot{p} + \frac{\partial \alpha}{\partial t} \tag{352}$$

$$\frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial \alpha}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial \alpha}{\partial t} \tag{353}$$

$$= [\alpha, H] + \frac{\partial \alpha}{\partial t} \tag{354}$$

which immediately shows that if H is not explicitly dependent on time then $\frac{dH}{dt} = 0$ *and is a constant of motion or a conserved quantity. Which also shows that if a variable* $\alpha(p, q)$ *is not explicitly a function of time i.e.* $\frac{\partial \alpha}{\partial t} = 0$ *and it's Poisson bracket with H i.e.* $[\alpha, H]$ = 0*, then* α *is a constant of motion. We can also prove that if two functions* α *and* β *are constants of motion i.e.* $\frac{d\alpha}{dt} = 0$ *and* $\frac{d\beta}{dt} = 0$ *then* $[\alpha, \beta]$ *is also a constant of motion.*

The fact that $[q_i, p_k] = \delta_{ik}$ *can also be used to check the canonical nature of a transformation because after transformation* $[Q_i, P_k] = \delta_{ik}$*. For example consider the generating function* $F_3(Q, p) = -(e^Q - 1)^2 \tan p$*. Then the transformation equations are given by;*

$$\implies q = -\frac{\partial F_3}{\partial p} = -(e^Q - 1)^2 \sec^2 p \tag{355}$$

$$\implies P = -\frac{\partial F_3}{\partial Q} = -2(e^Q - 1)e^Q \tan p \tag{356}$$

which may be de-convoluted to give

$$Q = ln(1 + \sqrt{q}\cos p) \tag{357}$$

$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p \tag{358}$$

Is the transformation canonical?

What about the transformation $P = q$ *and* $Q = p$

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = -1(\neq 0) \tag{359}$$

Since the above Poisson bracket is $\neq$ 1 *the transformation is not canonical! However, as seen before we can show that for the transformation* $P = -q$ *and* $Q = p$, $[Q, P]$ = 1 *i.e. the transformation is canonical.*

Consider the transformation and test whether it is canonical or not.

$$Q = -p\sin(t) + q\cos(t) \quad (360)$$
$$P = p\cos(t) + q\sin(t) \quad (361)$$
$$\Longrightarrow [Q, P] = \cos^2 t + \sin^2 t = 1 \quad (362)$$

which proves that the transformation is canonical. Obtain the generating function?

Consider the transformation below and derive the condition for the transformation to be canonical.

$$X = \frac{\alpha p}{x} \quad (363)$$
$$P = \beta x^2 \quad (364)$$
$$\Longrightarrow [X, P] = -2\alpha\beta \quad (365)$$
$$\Longrightarrow \beta = -\frac{1}{2\alpha} \quad (366)$$

if the generating function is of the type $F_1(x, X, t)$ *then;*

$$p = \frac{\partial F_1}{\partial x} = \frac{Xx}{\alpha} \quad (367)$$
$$\Longrightarrow F_1 = \frac{Xx^2}{2\alpha} + f_1(X) \quad (368)$$
$$P = -\frac{\partial F_1}{\partial X} = \beta x^2 \quad (369)$$
$$\Longrightarrow F_1 = -\beta X x^2 + f_2(x) = \frac{Xx^2}{2\alpha} + f_2(x) \quad (370)$$

together it shows that $F_1 = \frac{Xx^2}{2\alpha}$*. Now apply the above transformation to the case of a SHO and obtain the solution in terms of the transformed variables* P *and* X *and then the original variables* p *and* x*.*

Homework

1. *Setup the Hamiltonian for the following:*
   (i) a free particle in spherical polar coordinate system.
   (ii) the harmonic oscillator.
   (iii) a top whose Lagrangian is given by;

   $$L = \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}B(\dot{\psi}^2 + \dot{\phi}^2 \cos^2\theta) - V$$

   $\theta, \phi\psi$ are the generalized coordinates and A, B are constants and V is a function of coordinates only.
   (iv) a free particle in a rotating reference frame.

2. *The Hamiltonian of a system with 2 DOF is $H = \frac{1}{2}(p_1^2 q_1^4 + p_2^2 q_1^2 - 2\alpha q_1)$, where α is a constant. Show that q_1 varies sinusoidally with q_2.*

3. *Use Hamilton's formalism to solve for the equations of motion;*
   (i) $H = (1/2m)(p_r^2 + p_\theta^2/r^2) + V(r)$
   (ii) $H = (1/2m)(p_x^2 + p_y^2) + mgy$

4. *Investigate the motion of a particle whose Hamiltonian is $H = \alpha p^2 + \beta e^{q^2}$, where α and β are constants.*

5. *Consider a simple pendulum with the mass of the bob varying as a function of time as $m = m_0 e^{\alpha t}$. Write down the Lagrangian of the system and derive the EOMs. Obtain the Hamiltonian of the system from L. Calculate dH/dt and show that it is $= -\alpha L$.*

6. *Consider a modified Atwood machine as shown in the figure 47 The two blocks on the left have equal masses m and are linked by a massless spring of force constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$ where l_e is the equilibrium length (with all the weights in position and M held stationary).*

   (a) Show that the total potential energy is $V = \frac{1}{2}kx^2$ (plus a constant that can be ignored). (b) Find the two momenta conjugate to x and y and write down the Hamiltonian. Show that the coordinate y is ignorable. (c) Write down the four Hamilton equations and solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_0$. Still holding M fixed, you pull the lower mass m down a distance x_0, and at $t = 0$ you let go of both masses. Describe the motion. In particular, find the frequency with which x oscillates. [Hint: Write down the initial values of x, y and their momenta. You can solve the x equations by combining them into a second-order equation for x. Once you know $x(t)$, you can quickly write down the other three variables.]

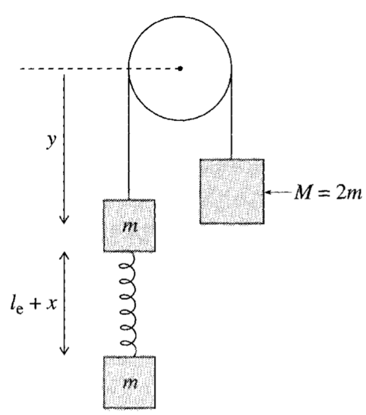


Figure 47: Modified Atwood's machine

7. *Consider a mass m confined to the x axis and subject to a force $F = kx$ where k > 0. (a) Write down and sketch the potential energy $V(x)$ and describe the possible motion of the mass. (Distinguish between the cases that $E > 0$ and $E < 0$.) (b) Write down the Hamiltonian, and describe the possible phase-space orbits for the two cases $E > 0$ and $E < 0$. (Remember that the function $H(x, p)$ must equal the constant energy E.) Explain your answers to part (b) in terms of those to part (a).*

8. *In discussing the oscillation of a mass at the end of a spring, we assume the spring is massless. Set up the Hamiltonian for a block of mass m attached to a spring (force constant k) whose mass M is not negligible, using the extension x of the spring as the generalized coordinate. Solve Hamilton's equations and show that the mass oscillates with angular frequency $\omega = \sqrt{k/(m + M/3)}$. That is, the effect of the non-zero spring's mass is to add $M/3$ to m. (Assume that the spring's mass is distributed uniformly and that it stretches uniformly.)*

9. *Show that the Hamiltonian for a charged particle (e) in an electromagnetic field is given by;*

   $L = \frac{1}{2}m\mathbf{v}^2 + e\mathbf{v}.\mathbf{A} - eV$

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + eV$$

   $\mathbf{A}$ *and V are the vector and scalar potentials respectively.*

10. *A Hamiltonian in a rotating coordinate system (constant angular velocity ω) is not the total energy, but nevertheless is a conserved quantity if the potential has rotational symmetry about the axis of rotation. Derive an expression for the difference between the H and total energy. Can you identify the result with a physical parameter of the system?*

11. *Find the relation between the constants a and b if the following transformation is to be canonical. $q = aP^{\frac{1}{2}} \sin Q, p = bP^{\frac{1}{2}} \cos Q$. Find the the values of a and b post transformation if the H of a harmonic oscillator is to be a multiple of P.*

12. *Find the canonical transformation by which the H for a freely falling particle becomes a multiple of P after the transformation.*

13. *For a harmonic oscillator $H = \frac{1}{2}(p^2 + \omega^2 q^2)$. If the generator function of a transformation is given by $F(q, Q) = \frac{1}{2}\omega q^2 \cot(2\pi Q)$, obtain the Hamiltons EOM with the transformed coordinates and solve for the same.*

14. *Find the form of the function $f(q)$ below for which the transformation will be canonical.*

$$Q = f(q) \cos p \qquad p = f(q) \sin p$$

15. *Consider a harmonic oscillator with* $H = \frac{1}{2}(p^2 + \omega^2 q^2)$.
(i) Derive the Lagrangian of the system from H.
(ii) Perform a double Legendre transformation on L as shown below to replace q and $\dot{q}$ *with p and* $\dot{p}$ *to derive the new function K.*

K is known as the momentum space Lagrangian.

$$K(p, \dot{p}, t) = L(q, \dot{q}, t) - p\dot{q} - q\dot{p} \quad (371)$$

(iii) Show that K has the same functional form in momentum space as the Lagrangian L has in the coordinate space.
(iv) Show that K satisfies the EL EOMs in the momentum space;

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{p}}\right) - \frac{\partial K}{\partial p} = 0$$

16. *Prove that for* $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$, *the function* $u(p, q, t) = ln(p + im\omega q) - i\omega t$ *is a constant of motion. Hint: show that* $\frac{du}{dt} = [\alpha, H] + \frac{\partial \alpha}{\partial t}$ *= 0.*

Rotational Reference Frames

- *The arbitrary motion of a point in space may always be instantaneously described as a pure rotation about an instantaneous axis of rotation i.e. as if the point is moving along the circumference of an imaginary circle, with the axis of rotation passing through the centre of the circle and* $\perp' r$ *to its plane*

- *The linear velocity of the particle at every instant may therefore be written as*

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{372}$$

  where $\vec{v}$ *is the linear velocity,* $\vec{\omega} = \frac{d\theta}{dt}\hat{n}$ *is the instantaneous angular velocity and* $\vec{r}$ *is the position vector of the point*

- *Successive finite rotation operations are non-commutative but infinitesimal rotations are commutative*

- *Infinitesimal rotations can be represented by axial vectors but finite rotations* <u>*cannot*</u> *be represented as vectors*

  Say a point moves such that it's position vector $\vec{r}$ *changes to* $\vec{r} + d\vec{r}$ *over time dt. Since this motion may be described by pure rotation the change is given by,*

$$d\vec{r} = d\vec{\theta} \times \vec{r} \tag{373}$$

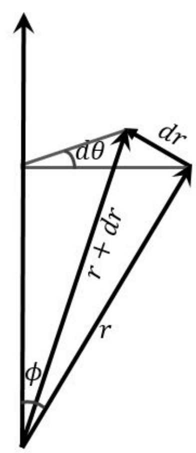


Figure 48: Instantaneous axis of rotation

Appendix A: D'Alembert's Principle: *Extending the principle of virtual work from static to dynamical systems.*

Virtual Displacement

Consider a mechanical system with the system coordinates $\vec{r}_i$, which are all not independent but related via constraints. Each coordinate undergoes virtual displacements $\delta\vec{r}_i$ subject to the following understanding.

1. *Time is held fixed during the virtual displacement, thus no actual displacement of the system takes place.*
2. *The virtual displacements do not violate the constraint equations, but may make use of any remaining unconstrained DOF*
3. *Are consistent with the forces imposed on the system*
4. *$\dot{\vec{r}}_i$ is the corresponding velocity*

Similarly virtual displacements of generalised coordinates δq_i necessarily satisfy all of the above.

Virtual Work

Using the virtual displacement, we may define virtual work δW as the work that would be done on the system by the forces acting on the system as the system undergoes the virtual displacement;

$$\delta W = \sum_i \vec{F}_i.\delta\vec{r}_i \tag{374}$$

where $\vec{F}_i$ ($\vec{F}_i = \vec{F}_i(nc) + \vec{f}_i(c)$) is the total force acting on the coordinate of the i^{th} particle and is the sum of external non constraint $\vec{F}_i(nc)$ and *constraint forces $\vec{f}_i(c)$. For a system in static equilibrium the total force on a constituent particle $\vec{F}_i$=0, therefore;*

$$\delta W = 0 \text{ (at equilibrium)}$$

Assumption: The net virtual work done by constraint forces is zero; *$\sum_i \vec{f}_i(c).\delta\vec{r}_i = 0$. Note: We DO NOT assume that the virtual work done by each constraint force is individually zero but the NET work done under virtual displacement is zero. This implies that at equilibrium;*

$$\boxed{\sum_i \vec{F}_i(nc).\delta\vec{r}_i = 0}$$

This is called **principle of virtual work**. *The net work done by the non-constraint forces is zero for virtual displacements, at static equilibrium. Can we have an equivalent deduction for dynamical systems? We'll answer that a bit later. First analyse the assumption introduced above and what kind of constraints satisfy the assumption. Schaum's Outline on Lagrangian Dynamics (D. A. Wells) says "While the truth of this statement is easily demonstrated with simple examples, a general proof is usually not attempted. It may be regarded as a postulate." Goldstein simply states that "We now restrict ourselves to systems for which the net virtual work of the forces of constraint is zero". Hand and Finch state "Recall that since constraint forces always act to maintain the constraint, they point in a direction perpendicular to the movement of the parts of the system. This means that the constraint forces do not contribute anything to the virtual work." But all these statements are either not obviously true or is confusing as to what they actually mean. Find an example where they are apparently violated.*

Problem X

Consider a pulley system as shown in figure 49 and determine its EOM. What are the constraint forces and the net virtual work done? Assuming the pulley is massless, the Lagrangian of the system is given by;

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + g(m_1 - m_2)x - m_2 gl$$

and the EOMs following the Euler Lagrange equation is;

$$(m_1 + m_2)\ddot{x} - g(m_1 - m_2) = 0,$$

As homework determine the constraint forces i.e. Tension and show that the net virtual work done is zero. Derive the Lagrangian and the EOM for the pulley system if the pulley is a disk of mass M and radius R. Show that the EOM is given by;

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} - g(m_1 - m_2) = 0,$$

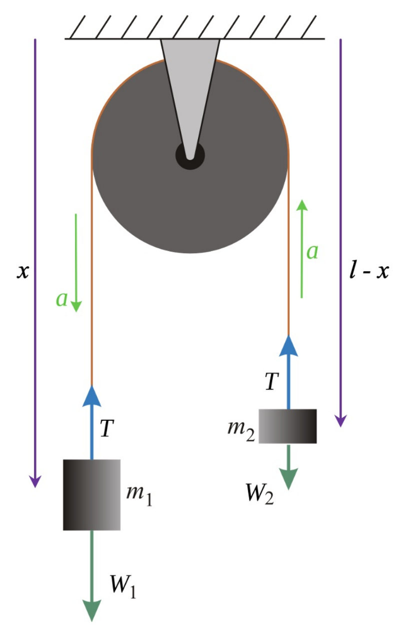


Figure 49: The pulley system: Atwood's machine

Now lets consider non-equilibrium, dynamical systems and see if there is an equivalent principle of virtual work. In dynamical systems Newton's second law states that;

$$\dot{\vec{p}}_i - \vec{F}_i = 0 \tag{375}$$

$$\implies \sum_i (\dot{\vec{p}}_i - \vec{F}_i).\delta\vec{r}_i = 0 \tag{376}$$

$$\sum_i (\dot{\vec{p}}_i - \vec{F}_i(nc)).\delta\vec{r}_i - \sum_i \vec{f}_i(c).\delta\vec{r}_i = 0 \tag{377}$$

Again, the last term vanishes invoking the clause that the net virtual work done by constraint forces is zero and we are left with;

$$\boxed{\sum_i (\dot{\vec{p}}_i - \vec{F}_i(nc)).\delta\vec{r}_i = 0} \tag{378}$$

This is known as D'Alembert's principle of virtual work.

Now since all the $\vec{r}_i$ may not be independent in general the corresponding $\delta\vec{r}_i$ are not necessarily independent either. Thus we cannot demand that their individual coefficients go to zero i.e. $\dot{\vec{p}}_i - \vec{F}_i(nc) \neq 0$. Note that physically, the rate of change of momentum of a particle ($\dot{p}$) is NOT equal to only the non-constraint forces $\vec{F}_i(nc)$, but the net force acting on it - Newton's 2^{nd} Law! However, if we transform the coordinates to a set of independent generalised coordinates we can demand that the coefficient of every δq_i be equal to zero. Consider the transformation that maps the $\vec{r}_i$ to a set of independent gerneralised coordinates q_i[19].

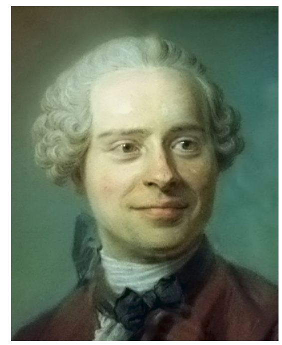


Figure 50: Jean d'Alembert (1717–1783). Traité de dynamique (1743)

[19] In writing the transformation equations we are assuming that the constraints are holonomic. Till the statement of D'Alembert's principle above the nature of the constraints is not relevant, but not hereafter

$$\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_n, t) \tag{379}$$

$$\implies \delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j \tag{380}$$

$$\dot{\vec{r}}_i = \vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \tag{381}$$

$$\implies \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \tag{382}$$

Now in terms of the generalised coordinates and their variation the virtual work done becomes;

$$\delta W = \sum_i \vec{F}_i.\delta\vec{r}_i = \sum_{i,j} \vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_j}\delta q_j \tag{383}$$

$$= \sum_j \mathcal{F}_j \delta q_j \tag{384}$$

$$\text{where } \mathcal{F}_j = \sum_i \vec{F}_i.\frac{\partial \vec{r}_i}{\partial q_j} \tag{385}$$

The $\mathcal{F}_j$ are known as generalised forces - but do not necessarily have the dimensions of force, since the q_i do not necessarily have the dimension of *distance. Now lets consider the first term of the equation 376;*

$$\sum_i \dot{\vec{p}}_i.\delta\vec{r}_i = \sum_i m\ddot{\vec{r}}_i.\delta\vec{r}_i = \sum_{i,j} m\ddot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \tag{386}$$

$$\sum_i m\ddot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_j} = \sum_i [\frac{d}{dt}(m\dot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_j}) - m\dot{\vec{r}}_i.\frac{d}{dt}\frac{\partial\vec{r}_i}{\partial q_j}] \tag{387}$$

$$\frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial q_j}) = \frac{\partial\dot{\vec{r}}_i}{\partial q_j} = \frac{\partial\vec{v}_i}{\partial q_j} \tag{388}$$

The last equation follows from interchange of time (d/dt) and coordinate ($\partial/\partial q_j$) derivatives. Note: This interchange is possible if the 2 differentiation operations commute when acting on $\vec{r}_i$, which is a non-trivial statement because q_i is time-dependent. Now lets go back to equation 376.

$$\sum_i(\dot{\vec{p}}_i - \vec{F}_i).\delta\vec{r}_i = \sum_i \dot{\vec{p}}_i.\delta\vec{r}_i - \sum_i \vec{F}_i.\delta\vec{r}_i = 0$$

$$\sum_{i,j}[\frac{d}{dt}(m\dot{\vec{r}}_i.\frac{\partial\vec{r}_i}{\partial q_j}) - m\dot{\vec{r}}_i.\frac{\partial\vec{v}_i}{\partial q_j}]\delta q_j - \sum_i \mathcal{F}_i\delta q_i = 0$$

$$\sum_{i,j}[\frac{d}{dt}(mv_i.\frac{\partial\vec{v}_i}{\partial\dot{q}_j}) - mv_i.\frac{\partial\vec{v}_i}{\partial q_j}]\delta q_j - \sum_i \mathcal{F}_i\delta q_i = 0$$

Using the fact that the kinetic energy $T = \sum m_i v_i^2/2$ and $\frac{\partial T}{\partial\dot{q}_i} = \sum mv_i.\frac{\partial\vec{v}_i}{\partial\dot{q}_j}$, we can write that

$$\sum_i[\frac{d}{dt}(\frac{\partial T}{\partial\dot{q}_i}) - \frac{\partial T}{\partial q_j} - \mathcal{F}_i]\delta q_i = 0 \tag{389}$$

This is finally the D'Alembert's principle for variation of generalised coordinates, which are all independent and arbitrary. Thus the term in square brackets must each individually go to zero.

$$\frac{d}{dt}(\frac{\partial T}{\partial\dot{q}_i}) - \frac{\partial T}{\partial q_j} = \mathcal{F}_i \tag{390}$$

For forces derivable from a scalar potential V, the generalised force is given as $\mathcal{F}_i = -\partial V/\partial q_i$ and the above equation can be rewritten as;

$$\frac{d}{dt}(\frac{\partial(T-V)}{\partial\dot{q}_i}) - \frac{\partial(T-V)}{\partial q_j} = 0 \tag{391}$$

The term $T - V = L$ is the Lagrangian of the system and the equation is akin to the Euler Lagrange equations of motion. In conclusion, Note the following;

1. *The general form of D'Alembert's principle does not assume anything about the nature of the constraints.*

2. *writing the coordinate transformation equation then assumes the constraints to be holonomic*

3. *Identification of the Lagrangian to be of the form* $L = T - V$ *along with the above EOM is valid only when the forces are derivable from a potential and the constraints are holonomic.*

Can you deduce the above for a system with variable mass?

Appendix B: Hidden Symmetry of the Laplace Runge Lenz Vector

Apart from the facts that the inverse square law force allows closed orbits that are conic sections like ellipses, circles and precessing ellipses it is also characterised by a large number of conserved quantities. In addition to the obvious conservation of energy E and angular momentum L, the Kepler problem yields an additional conserved quantity namely the Laplace-Runge-Lenz (LRL) vector. Following Noether's theorem of associating conserved quantities with symmetry transformations one wonders what symmetry the LRL vector is associated with.

By choosing a nice parametrization we show that the equations of motion and the conservation of energy describe a harmonic oscillator with an extra derivative in four dimensions and a four dimensional sphere, respectively. From this we define a conserved bivector. The components of this bivector correspond to the LRL vector and angular momenta The Kepler problem concerns the motion of 2 bodies interacting via a inverse square law force or the equivalent one-body problem with reduced mass m and position vector $\vec{r}$*, moving about an attractive centre of force at the origin.*

$$F = \frac{-k}{r^2}\hat{r} \tag{392}$$

where k is a constant. The EOM is given by,

$$m\frac{d^2\vec{r}}{dt^2} = \frac{-k}{r^2}\hat{r} \tag{393}$$

and the energy conservation relation given as,

$$\frac{m}{2}(\frac{d\vec{r}}{dt})^2 - \frac{k}{r} = E \tag{394}$$

Elliptical case

The $E < 0$ case yields bound states with elliptical orbits. And we can define some constants-

$$V = \sqrt{\frac{-2E}{m}}$$
$$R = \frac{-k}{2E} \quad (395)$$
$$T = \frac{R}{V}$$

here we now consdier the keppler problem is space time where time is no more a parametre but become a coordinate! becuase of this we are going to use some arbitary parametre S instead of ususal time t. Differentiating with respect to S will be denoted by prime. We demand that this new parameter is such that the following equalities hold:

$$\frac{1}{t'} = \frac{dS}{dt} = \frac{V}{r}$$
$$\text{and } r' = \frac{dr}{dS} \quad (396)$$

See-this new kind of time ticks more slowly as you get farther from the sun. So, using this new time speeds up the planet's motion when it's far from the sun. If that seems backwards, just think about it. For a planet very far from the sun, one day of this new time could equal a week of ordinary time. So, measured using this new time, a planet far from the sun might travel in one day what would normally take a week

now using (5) we can modify (3) as-

$$V^2(t' - T)^2 + |\vec{r'}|^2 = R^2 \quad (397)$$

and this is a equation of spehre in 4D space!!(equation of a 4D spehre)

Since the right hand side of the defining constraint $t' = \frac{r}{V}$ has dimension of time, it follows that differentiating with respect to S does not change the dimension and so S must be a dimensionless quantity. This leads to that although r' and r'' are to be regarded as a kind of velocity and acceleration vectors, they have the dimension of length and are valid space vectors. In particular, r, r' and r'' can be added or subtracted from each other.

Lets simplify the equation (6) by assuming m=1,E=-1/2,k=1

$$(t' - 1)^2 + |\vec{r'}|^2 = 1 \quad (398)$$

So what is ***means?***

the point (t,x,y,z) in space time cooridiante moves around in 4-dimensional space as the parameter S changes. What we're seeing is that the velocity of this point, namely

$$v{=}(t', x', y', z')$$

moves around on a sphere in 4-dimensional space! It's a sphere of radius one centered at the point (1,0,0,0) with some further calculation we can show that

$$r''' = -r'$$
$$t''' = -(t' - 1) \tag{399}$$

We can state both of them in words as follows: the 4-dimensional velocity v carries out simple harmonic motion about the point (1,0,0,0)

That's nice. But since v also stays on the unit sphere centered at this point, we can conclude something even better: v must move along a great circle on this sphere, at constant speed! (for a rough visualization in 3D click on this link)

This implies that the spatial components of the 4-dimensional velocity have mean 0 , while the t component has mean 1

The first part here makes a lot of sense: our planet doesn't drift ever farther from the Sun, so its mean velocity must be zero. The second part is a bit subtler, but it also makes sense: the ordinary time t moves forward at speed 1 on average with respect to the new time parameter S, but its rate of change oscillates in a sinusoidal way.

Conserved space time bivector

The cross product is actually valid only in 3D. whats it generalization in higher dimensional space? (This will help to understand bivector through application)Thats why we are introducing the concept of bivector. Since v-curves are great circles with constant speed we have that $\tau = v \wedge v'$ is a conserved spacetime bivector in 4D.(where $v = (t' - 1)e_t + \vec{r'}$). It can be computed as-

$$\tau = e_t \wedge -\frac{1}{r}((r-1)\vec{r} + r'\vec{r'}) + \frac{1}{r}\vec{r} \wedge \vec{r'} \tag{400}$$

It consist of a conserved spatial bivector $L = \frac{1}{r}\vec{r} \wedge \vec{r'}$ and a conserved space vector $A = -\frac{1}{r}((r-1)\vec{r} + r'\vec{r'})$.The great circles are projected onto the spatial subspace as centered ellipse-

Such an ellipse lies in the plane given by the bivector L and has major semiaxis R and area $\pi|L|$. With the help of the vector A the equation of motion for r'' can be expressed as(see for detail):

$$r'' = -r + A \tag{401}$$

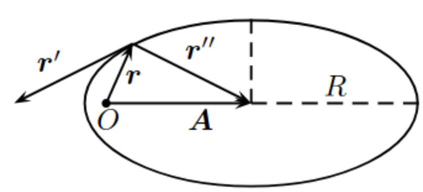


Figure 51

We can show that the conserved bivector L can be identified with the angular momentum bivector,and the the conserved vector A corresponds to the so called Laplace-Runge-Lenz vector.

To summarize, we saw that this parametrization in space time gives the equa- tions of motions and conservation of energy a nice form. From these *equations the symmetry group SO(4) becomes apparent. These equations also provide a way of defining the angular momentum and the Runge-Lenz vector There are many more ways to reach the same conclusion as mentioned-*

1.*Algebric approach: -algebric approach*

2.*Classical approach:classical approach using stereographic projection.*

3.*The Kepler problem and Jordan algebras*

4.*The Kepler problem and supersymmetry*

References

1.*Göransson's work*

2.*a webpage with animation and all*

3.*a webpage about velocity circles and momentum space projection*

4.*A paper containing both algebric and analytic approach*

5.*detail proof via the stationary action principle, by Norcliffe and Percival, that the path on S^3 is a great circle*

6.*bivector*

7.*D.hestenes,New foundation for Classical mechanics*

8.*http://staff.ustc.edu.cn/ bjye/LX/LRL.pdf*