JOY MITRA

ELECTROMAGNETISM

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preface and a disclaimer

These notes have been used to teach the Electromagnetism to the first year BS-MS students at the School of Physics, IISER TVM, in the Vasant semesters of 2011, 2012, 2013, 2021). They combine material from standard books on the subject like Griffiths, Purcell, Feynman Lectures and some selected material from Jackson, Panofsky & Phillips, etc.. But this compilation is my take on teaching Electromagnetism and thus different from each of these texts individually, in approach. The PhET Interactive Simulations, made available by the University of Colorado, Boulder at https://phet.colorado.edu were of immense help while teaching this course online during the COVID-19 pandemic of 2020 - 2021. The simulations made a significant difference to the way in which concepts could be visualised and communicated to the audience. Snapshots from the simulations are also included in the notes. As of now I do not have typed version of the notes but are handwritten, which I hope to translate into a typed version sometime in future.

These notes are likely riddled with multiple errors and typos - so be forewarned! These notes are not a substitute for proper textbooks and should be read in conjunction with them.

books

1. Introduction to Electrodynamics (4th Ed), David J. Griffiths
2. Electricity and Magnetism (Berkeley Physics Course Volume II), Edward M. Purcell
3. Feynman Lectures (Millennium Ed.), Feynman, Leighton, Sands
4. Physics for Scientists and Engineers, Serway and Jewitt
5. Classical Electrodynamics, J D Jackson
6. Panofsky and Phillips

Chapter 0 Beyond Mechanics

Beyond Mechanics

Laws of Nature

1. Gravity: $\bar{F} = -G \dfrac{M_1 M_2}{r^2} \hat{r}$ $\qquad G = 6.67430 \times 10^{-11}\ m^3/s^2 kg$

$N\,m^2/Kg^2$

2. Electrostatics: $\bar{F} = \dfrac{1}{4\pi\epsilon_0} \dfrac{Q_1 Q_2}{r^2} \hat{r}$ $\qquad \dfrac{1}{4\pi\epsilon_0} \simeq 9 \times 10^9\ N\,m^2/C^2$

$N\,m^2/A^2 s^2$

3. Magnetism: $\bar{F} = \dfrac{\mu_0}{4\pi} \dfrac{P_1 P_2}{r^2} \hat{r}$ $\qquad \dfrac{\mu_0}{4\pi} = 10^{-7}\ H/m$

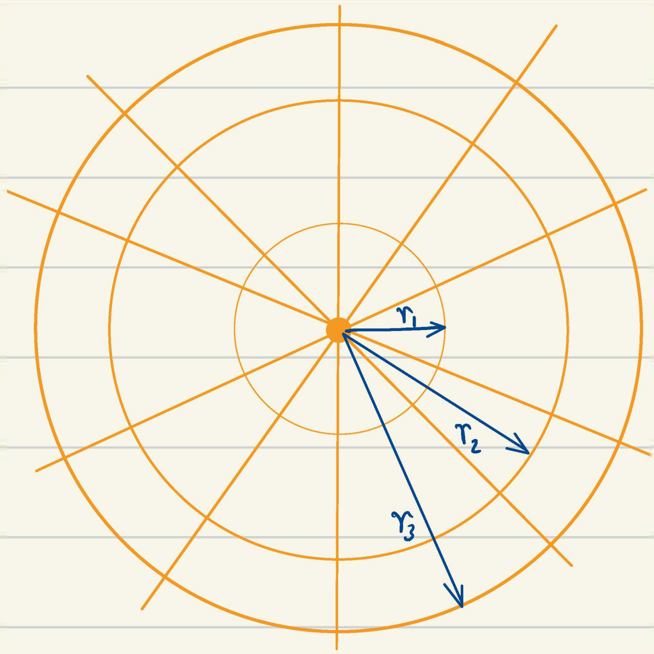
All these laws are inverse square

Involve fundamental constants G, ϵ_0, μ_0 ....

Why are they $1/r^2$? Why do the constants have the values they have?

Are all the constants fundamental?

Consider a point source of light that emits K photons isotropically per unit time



3 imaginary spheres of radii r_1, r_2, r_3 centred at the source.

$\boxed{\text{Calculate the Intensity on the 3 spheres!}}$

Intensity is the power transferred per unit area.

$$\left.\begin{aligned} \text{On } r_1: \quad I_1 &= \frac{K}{4\pi r_1^2} \\ r_2: \quad I_2 &= \frac{K}{4\pi r_2^2} \\ r_3: \quad I_3 &= \frac{K}{4\pi r_3^2} \end{aligned}\right\}$$

Anything that spreads out uniformly in our 3D world has a $1/r^2$ dependence.

$\sqrt{\epsilon_0 \mu_0} = 1/c$

ϵ_0 : permittivity of free Space
μ_0 : permeability of free Space
c : velocity of light

$$\alpha = \frac{e^2}{4\pi\hbar}\sqrt{\frac{\mu_0}{\epsilon_0}}$$

$\hbar$: Plank's constant $/2\pi$
m_e : mass of an electron

α : fine Structure constant
$\simeq 1/137 \longrightarrow$ Quantifies strength of EM Interactions between elementary particles

Today Newton's Law of Gravity replaced by Einstein's General Theory of Relativity.

G in Newtonian Gravity : Proportionality const
G in Einstein Relativity : Geometry of space-time + Energy momentum

$$G = \frac{4}{3}\frac{\hbar c}{m_e^2}\alpha^{21}$$

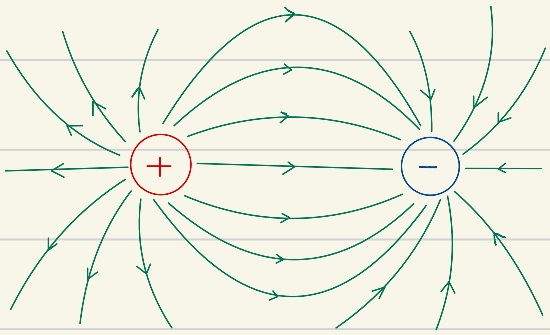
$\leftarrow$ An approximate relation from the TOE

Classical Electrodynamics: 17th – 19th C

Wikipedia page on
Timeline of EM and Classical optics

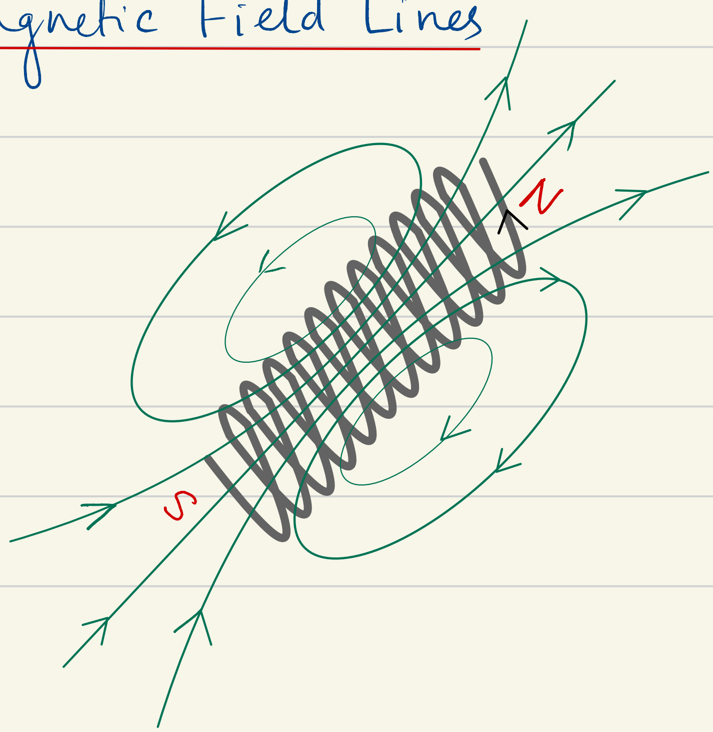
- Robert Boyle
- Benjamin Franklin
- Michael Faraday
- Fredrick Gauss
- Biot Savart
- Cavendish
- Oersted
- Ohm
- Weber
- Ampere
- Poisson
- Galvani
- Volta
- Coulomb
- Maxwell

Intuition Vs. Mathematical Rigor (Proof)



Electric Field lines

Magnetic Field Lines



But what is the reality of these field lines?

Gravity vs. Electrostatics

Ex: Calculate the ratio of electrostatic force of repulsion to gravitational attraction between 2 protons separated by a distance "d".

$$F_G = G\frac{m_1 m_2}{d^2} = 6.674\times10^{-11} \times \frac{(1.67\times10^{-27})^2}{d^2} \simeq \frac{18.6\times10^{-65}}{d^2}$$

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{d^2} = 9\times10^{9} \times \frac{(1.6\times10^{-19})^2}{d^2} \simeq \frac{23.0\times10^{-29}}{d^2}$$

ratio: $\frac{F_G}{F_E} = \frac{18.6\times10^{-65}}{23.0\times10^{-29}} \simeq 0.8\times10^{-36}$

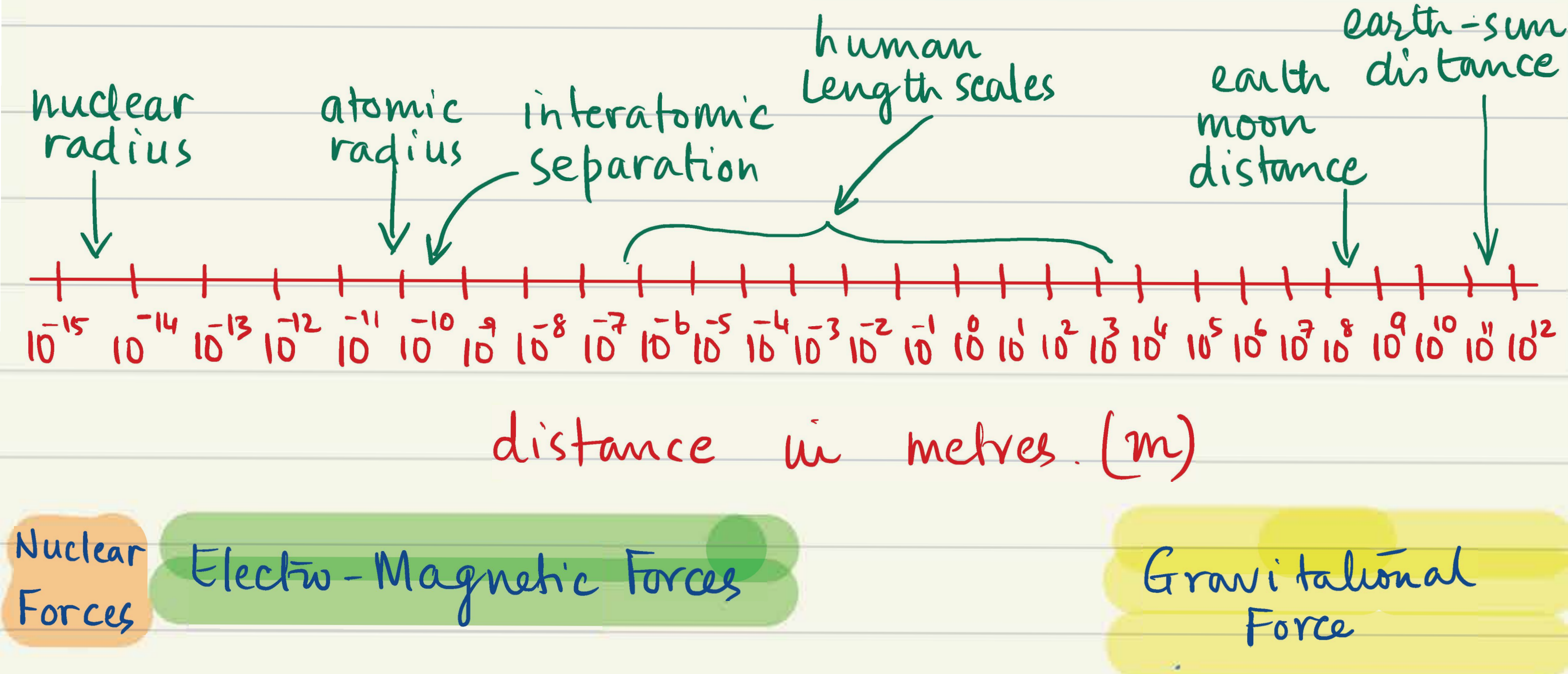
Gravitational Force $\ll$ Electrostatic force.

How do the protons stay together in the nucleus of atoms?

$r_{nucleus} \sim 10\times10^{-15}\,m$

Not Electrostatic Forces

The answer is the nuclear forces
The weak + strong interactions !



The 3 Fundamental constants of modern physics G, c, $\hbar$ can be combined to yield a length

Planck Length $l_P = \sqrt{\dfrac{\hbar G}{c^3}} \simeq 1.616 \times 10^{-35}$ m

we leave the implications of l_P for more advanced treatment

Homework:

1. List Biological / Chemical / Physical Systems/phenomena that are size dependent at various length scales, like the examples discussed above.

2. Plot the function $y = x^x$ and its derivative. Calculate $\lim_{x \to 0} x^x = ?$

Chapter 1 Vector Algebra and Vector Calculus

① Vector Algebra and Vector Calculus

→ A vector is a quantity that has both magnitude and direction.

Represented by an arrow $\vec{A}$  Orientation

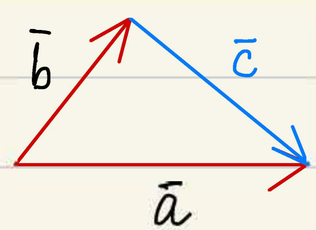
length ≡ magnitude $|\vec{A}|$

Need a proper Co-ordinate system to define the vectors.

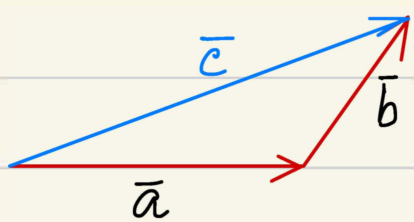
→ A quantity that has magnitude but no direction : Scalar

e.g. Temperature, mass

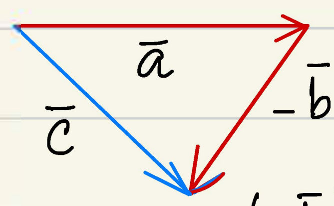
→ Vector Algebra



$\bar{a} = \bar{b} + \bar{c}$



$\bar{a} + \bar{b} = \bar{c}$

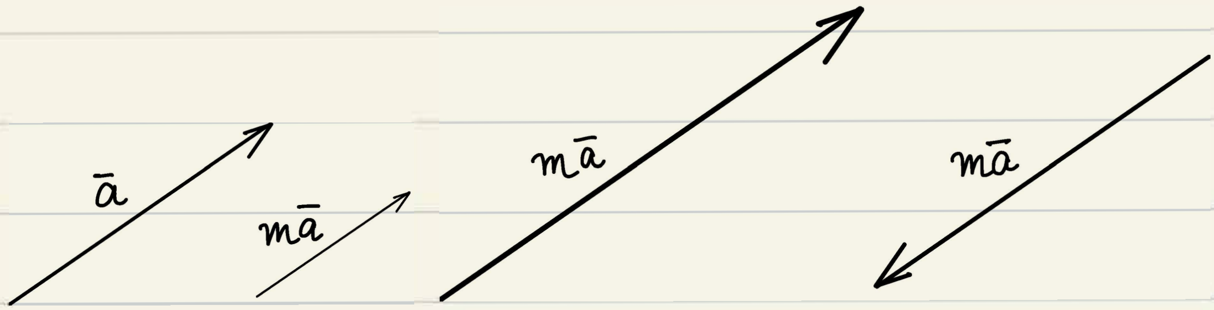


$\bar{c} = \bar{a} + (-\bar{b})$

$= \bar{a} - b$

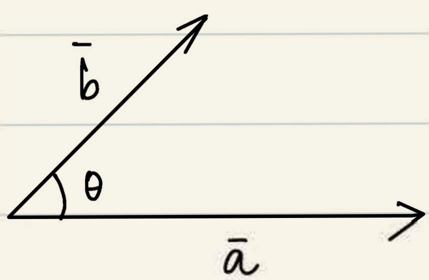
→ Multiplication of vectors

① <u>Scalar multiplication</u> $\bar{b} = m\bar{a}$ m is a scalar



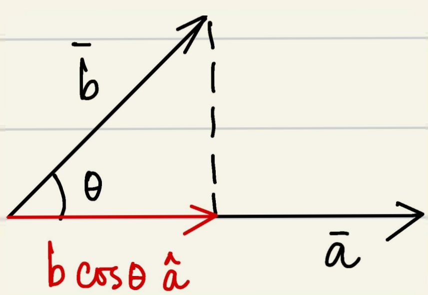
$0 < m < 1$ $m > 1$ $m < 0$

② <u>Dot Product</u>



$$\bar{a}.\bar{b} = \underbrace{|\bar{a}|\,|\bar{b}|\cos\theta}_{\text{scalar}}$$

$\boxed{\text{Scalar product}}$



↑
projection of $\bar{b}$ along $\bar{a}$

$$\bar{a}.\bar{b} = \begin{Bmatrix} \bar{a}\,.\,b\cos\theta\,\hat{a} \\ a\cos\theta\,\hat{b}\,.\,\bar{b} \end{Bmatrix} = ab\cos\theta$$

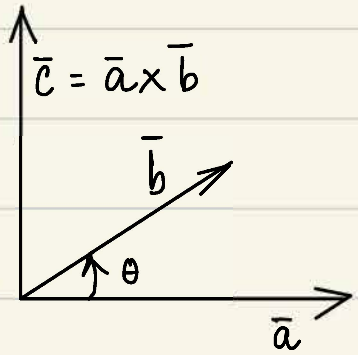
$$\Rightarrow \quad \bar{a}.\bar{b} = \bar{b}.\bar{a}$$

$\hat{a}/\hat{b}$: unit vector along $\bar{a}/\bar{b}$

$\bar{a}.\bar{b}$ is max. for $\theta = 0$ $\bar{a}.\bar{b} = ab\cos 0 = ab$

$\bar{a}.\bar{b}$ is min for $\theta = \pi/2$ $\bar{a}.\bar{b} = ab\cos\pi/2 = 0$

③ Cross Product ($\boxed{\text{Vector Product}}$)



$\bar{c} = \bar{a} \times \bar{b}$ ← Right Hand rule

$|\bar{c}| = |\bar{a}|\,|\bar{b}| \sin\theta$

$\Downarrow$

$\bar{c} \perp \bar{a}$ & $\bar{b}$

$\bar{a} \cdot (\bar{a} \times \bar{b}) = 0$

$\bar{b} \cdot (\bar{a} \times \bar{b}) = 0$

$\Rightarrow \bar{a} \times \bar{a} = 0$

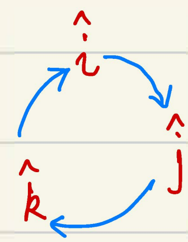
$\bar{a} \times \bar{b} \neq \bar{b} \times \bar{a}$

$\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

Cartesian unit vectors: $\hat{i}, \hat{j}, \hat{k}$

$\hat{i} \times \hat{i} = 0$ $\quad$ $\hat{i} \times \hat{j} = \hat{k}$ $\quad$ $\hat{j} \times \hat{k} = \hat{i}$ $\quad$ $\hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$ $\quad$ $\hat{k} \times \hat{j} = -i$ $\quad$ $\hat{i} \times \hat{k} = -\hat{j}$



What is the physical interpretation of $\hat{i} \times \hat{j}$?

Vector Identities

$\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

$\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

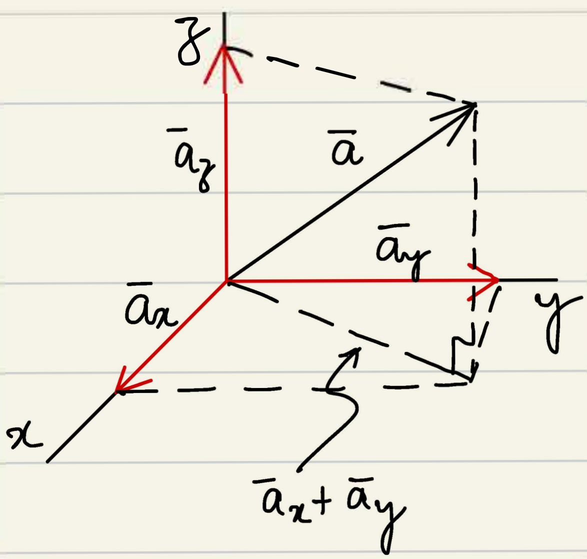
Components of Vectors

Vectors are expressed according to Co-ordinate Systems

3D physical world ⇒ 3 independent co-ordinates

Right handed system ← convention

① Cartesian Co-ordinates (x, y, z)



$\bar{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$

$|\bar{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

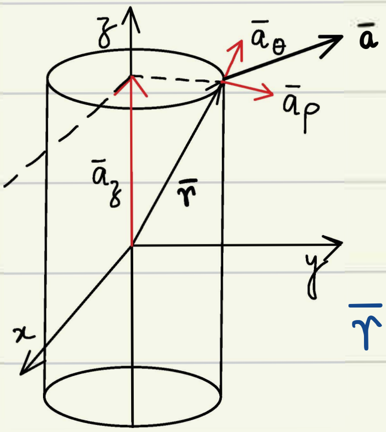
magnitude

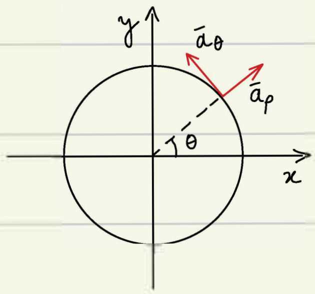
$(\hat{x}, \hat{y}, \hat{z}) \longrightarrow$ unit vectors along x, y, z

$(\hat{i}, \hat{j}, \hat{k})$

$\hat{a}$ is a unit vector along $\bar{a}$; $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$

② Cylindrical Co-ordinates (ρ, θ, z)





$$\bar{a} = a_\rho \hat{\rho} + a_\theta \hat{\theta} + a_z \hat{z}$$
$$= \underbrace{\bar{a}_\rho + \bar{a}_\theta}_{\text{projection on } xy \text{ plane}} + \underbrace{\bar{a}_z}_{\text{projection along } \hat{z}}$$

$$\bar{r} = \rho \hat{\rho} + z \hat{z}$$

↳ projection of $\bar{r}$ in xy plane

Note : $\hat{\rho}$ and $\hat{\theta}$ change direction at each point

$$\boxed{\begin{aligned} x &= \rho \cos\theta \\ y &= \rho \sin\theta \\ z &= z \end{aligned}} \Rightarrow \boxed{\begin{aligned} \rho &= \{x^2 + y^2\}^{1/2} \\ \theta &= \tan^{-1} y/x \\ z &= z \end{aligned}}$$

$$\vec{\rho} = x\hat{i} + y\hat{j} \Rightarrow \boxed{\hat{\rho} = x/\rho \, \hat{i} + y/\rho \, \hat{j}\,;}$$

Say;

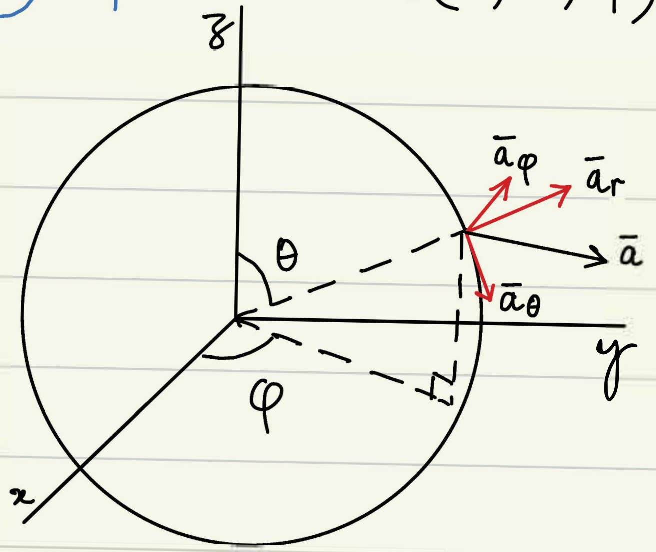
$$\hat{\theta} = a\hat{i} + b\hat{j} \qquad \begin{cases} \hat{\theta} \cdot \hat{\rho} = 0 \Rightarrow a x/\rho + b y/\rho = 0 \\ \hat{\rho} \times \hat{\theta} = \begin{vmatrix} i & j & k \\ x/\rho & y/\rho & 0 \\ a & b & 0 \end{vmatrix} = \left(\dfrac{xb}{\rho} - \dfrac{ya}{\rho} \right) \hat{k} \end{cases}$$

$$\left(\frac{xb}{\rho} - \frac{ya}{\rho} \right) = 1$$

$$a = -y/\rho \quad \text{and} \quad b = x/\rho$$

$$\Rightarrow \boxed{\hat{\varphi} = -\frac{y}{\rho}\hat{i} + \frac{x}{\rho}\hat{j}}$$

③ Spherical Polar (r, θ, φ)



unit vectors along r, θ, φ

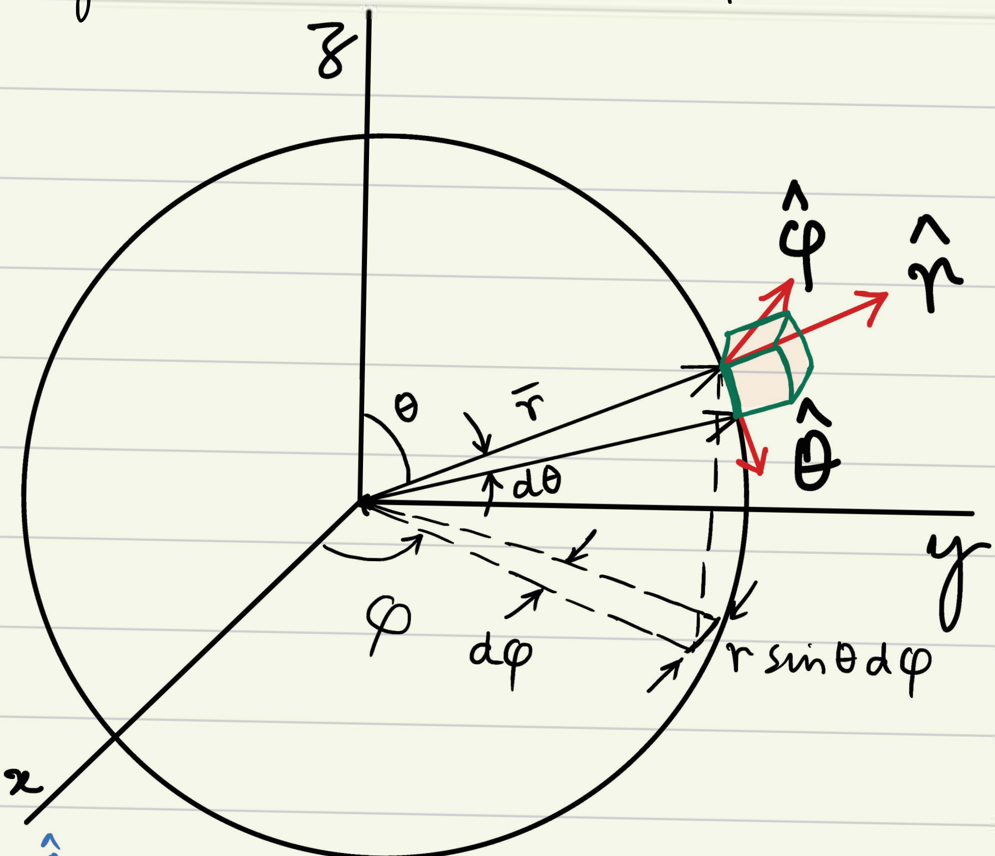
$$\bar{a} = a_r \hat{r} + a_\theta \hat{\theta} + a_\varphi \hat{\varphi}$$

$$\bar{a}_r + \bar{a}_\theta + \bar{a}_\varphi$$

the position vector $\bar{r}$

θ : polar angle

φ : azimuthal



Right handed System

$\hat{r} \times \hat{\theta} = \hat{\varphi}$ $\quad \hat{\theta} \times \hat{\varphi} = \hat{r}$ $\quad \hat{\varphi} \times \hat{r} = \hat{\theta}$

$$\boxed{\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= \underbrace{r \cos\theta}_{\text{projection along } z \text{ axis}} \end{aligned}}$$

$$\boxed{\begin{aligned} r &= \{x^2 + y^2 + z^2\}^{1/2} \\ \theta &= \tan^{-1} \frac{\{x^2 + y^2\}^{1/2}}{z} \\ \varphi &= \tan^{-1} y/x \end{aligned}}$$

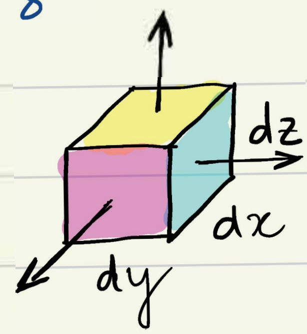
HOME WORK!

$\boxed{\text{Determine } \hat{r}, \hat{\theta}, \hat{\varphi} \longrightarrow \hat{i}, \hat{j}, \hat{k}}$

Cartesian:

Line element: $d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Area Element: $d\bar{a} = \begin{matrix} dx\,dy\,\hat{k} \\ dy\,dz\,\hat{i} \\ dz\,dx\,\hat{j} \end{matrix}$



Volume Element $dV = dx\,dy\,dz$

Cylindrical:

Line Element : $d\bar{r} = d\rho\,\hat{\rho} + \rho\,d\theta\,\hat{\theta} + dz\,\hat{z}$

Area Element : $d\bar{a} = \begin{matrix} \rho\,d\theta\,dz\,\hat{\rho} & \rho \text{ constant} \\ d\rho\,dz\,\hat{\theta} & \theta \text{ Constant} \\ \rho\,d\theta\,d\rho\,\hat{k} & z \text{ Constant} \end{matrix}$

Volume Element : $dV = d\rho\,\rho\,d\theta\,dz$

Spherical:

Line Element : $d\bar{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$

Area Element : $d\bar{a} = \left.\begin{matrix} r^2\sin\theta\,d\theta\,d\varphi\,\hat{r} & : r \text{ const} \\ r\,d\theta\,dr\,\hat{\varphi} & : \varphi \text{ Const} \\ r\sin\theta\,d\varphi\,dr\,\hat{\theta} & : \theta \text{ const} \end{matrix}\right\}$

Volume Element : $dV = r^2\sin\theta\,d\theta\,d\varphi\,dr$

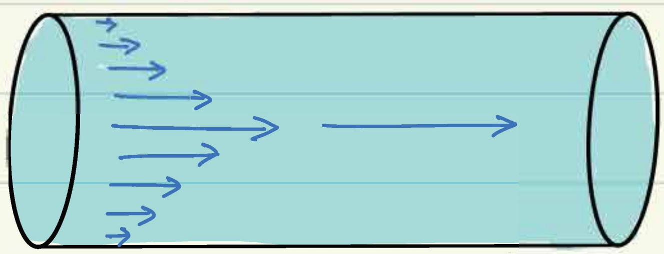
Vector Calculus

$T(x, y, z)$: Temperature at all points in a region

Scalar Field

$\rho(x, y, z)$: Mass Density at all points in a region

$\bar{F}(x, y, z)$: Velocity at all points in a fluid



velocity at the boundary = 0 m/s

Vector Field

$\vec{E}(x, y, z)$: Electric Field around a charge distribution.

→ Gradient of a Scalar Field

Consider a Function of a single variable; $f(x)$ $x \in$ all space

Change in f when $x \to x + \Delta x$ $\boxed{\Delta f = f(x+\Delta x) - f(x) = \frac{df}{dx} \Delta x}$

If f is a multivariable fn $f(x, y, z)$ then

the value of the function at $f(x+dx, y+dy, z+dz)$

$$= f(x, y, z) + \left\{ \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right\}$$

partial derivaties wrt indep. var

Change in f

$$df = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \cdot \left(dx\, \hat{i} + dy\, \hat{j} + dz\, \hat{k} \right)$$

$$= \bar{\nabla} f(x, y, z) \cdot d\bar{r}$$

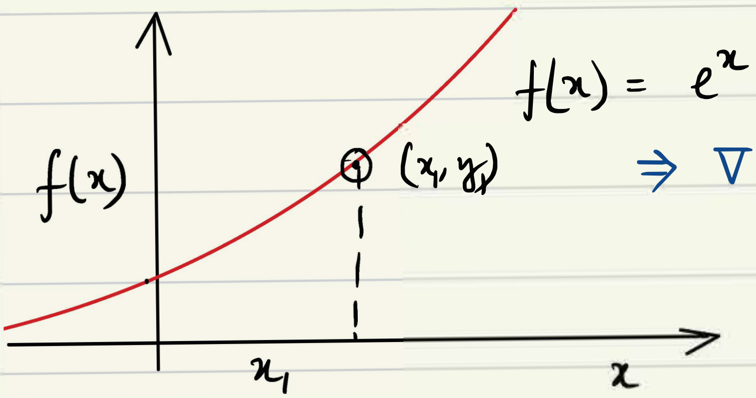
↓

vector ← gradient of a scalar

$\bar{\nabla} f \rightarrow$ Vector

- magnitude — measure of local rate of change
- direction — along highest change

at a point

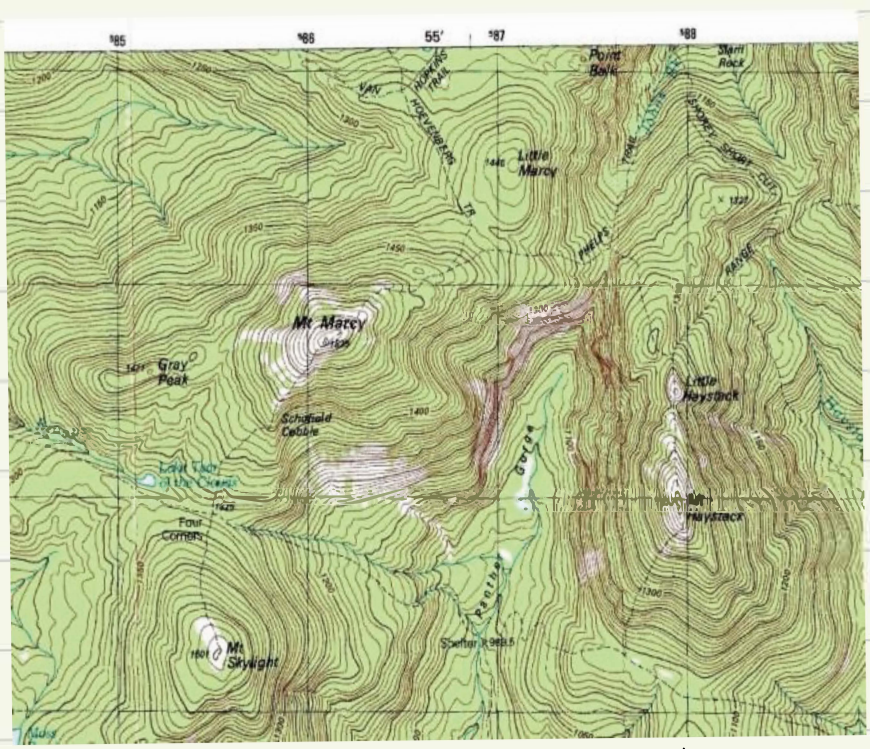


$f(x) = e^x$

$$\Rightarrow \nabla f(x_1) = \left.\frac{\partial f}{\partial x}\right|_{x_1} \hat{i} = e^{x_1}\hat{i}$$

↓

tangent



Topography Map of Heights

→ lines of constant heights $h(x, y)$

→ Dense lines denotes bigger changes over shorter length sc.

⇒ higher gradient

→ ∇h points along steepest change.

Eg: • $r = \{x^2 + y^2 + z^2\}^{1/2}$

$$\bar{\nabla} r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{1}{r}\left\{x\hat{i} + y\hat{j} + z\hat{k}\right\} = \frac{\bar{r}}{r} = \hat{r}$$

• $$\bar{\nabla}\left(1/r\right) = -\left[\frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k}\right] = \frac{-\bar{r}}{r^3} = \frac{-\hat{r}}{r^2}$$

• $f(x, y, z) = e^x \sin y \ln z$

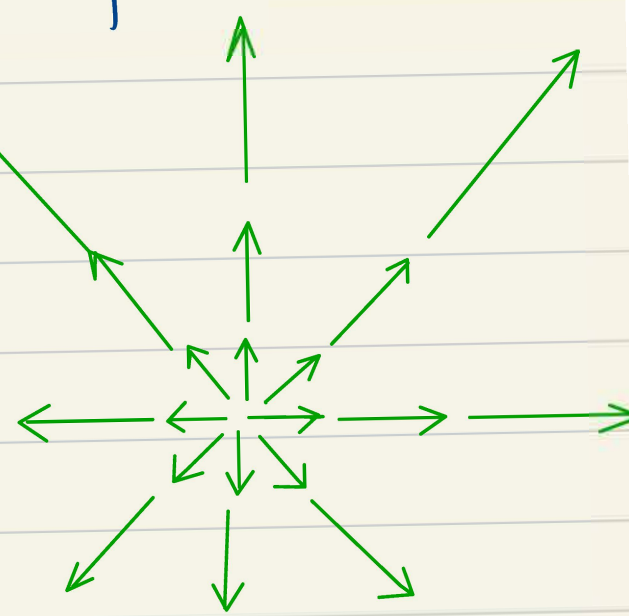
$$\vec{\nabla} f = e^x \sin y \ln z\,\hat{i} + e^x \cos y \ln z\,\hat{j} + e^x \sin y \frac{1}{z}\hat{k}$$

Divergence of a Vector

$$\bar{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \text{scalar}$$

Physically the divergence of a vector is a measure of how much it "spreads" at a point.

$\Rightarrow$ denotes a SOURCE



Examples:

- $\bar{v} = \alpha(x\hat{i} + y\hat{j} + z\hat{k})$

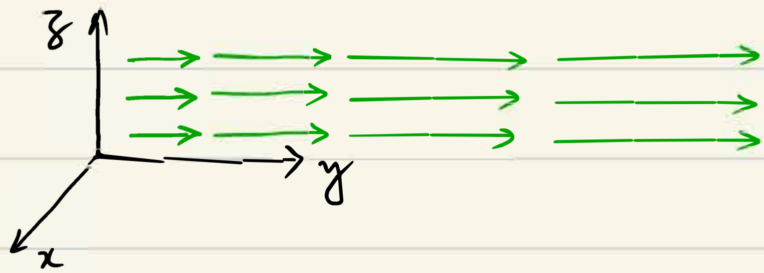
$$\nabla.\bar{v} = \alpha\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)$$

$$= 3\alpha$$

+ve $\bar{\nabla}.\bar{A}$ $\Rightarrow$ source

−ve $\bar{\nabla}.\bar{A}$ $\Rightarrow$ sink

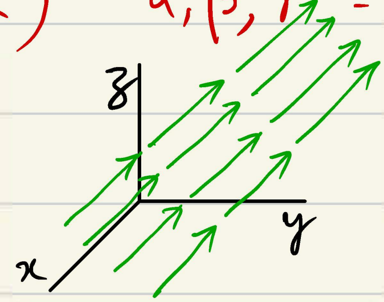
- $\bar{v} = \alpha y \hat{j}$



$\bar{\nabla} \cdot \bar{v} = \alpha$

Divergent Field of magnitude α

- $\bar{v} = (\alpha \hat{i} + \beta \hat{j} + \gamma \hat{k})$ α, β, γ = constants



$\nabla \cdot \bar{v} = 0$

- $\bar{v} = \hat{r}/r^2$

$$\bar{v} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \bar{\nabla} \cdot \bar{v} = \frac{\partial}{\partial x}\left(\frac{x}{(\ )}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(\ )}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(\ )}\right)$$

You can show $\bar{\nabla} \cdot \dfrac{\hat{r}}{r^2} = \dfrac{0}{(x^2+y^2+z^2)^{5/2}} = \dfrac{0}{r^5}$

$\Rightarrow \bar{\nabla} \cdot \dfrac{\hat{r}}{r^2} = 0$ at all points!

But what happens at $r = 0$?

Curl of a Vector Field

$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

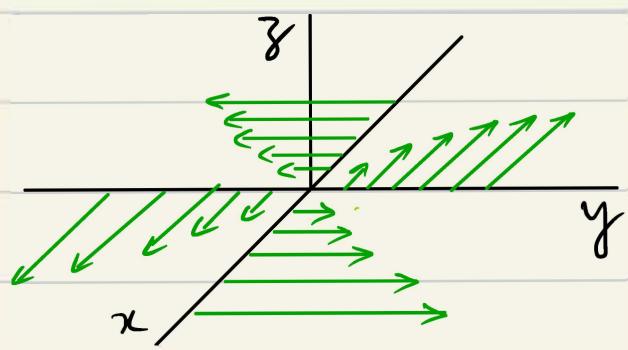
→ $\bar{\nabla} \times \bar{A}$ points ⊥'r to $\bar{A}$ at that point

→ $\bar{\nabla} \times \bar{A}$ is a measure of how much it "bends" or "curls" around at that point

→ Work out the curl of all the previous vector fields and show $= 0$.

- $\bar{v} = -y\hat{i} + x\hat{j}$

$\bar{\nabla} \times \bar{v} = 2\hat{k}$



The velocity distribution in a a whirlpool or twister are ideal examples of fields with "curl". Nothing "spreads" but they all go round in circles.

A garden sprinkler
→ source
→ divergent field

What about a rotating garden sprinkler?

Homework: Write the velocity distribution for that!

$\nabla \times \bar{v} \neq 0 \quad + \quad \bar{\nabla} \cdot \bar{v} \neq 0$

- Can we construct a vector field such that $\bar{\nabla} \times \bar{v} = 0$ and $\bar{\nabla} \cdot \bar{v} = 0$ but $\bar{v} \neq$ constant

$$\bar{\nabla} \cdot \bar{v} = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\bar{\nabla}\times\bar{v} = 0 \quad \Rightarrow$$

$$\frac{\partial v_z}{\partial y} = \frac{\partial v_y}{\partial z} = \alpha_1 \quad \Rightarrow \quad v_z = \alpha_1 y \quad ; \quad v_y = \alpha_1 z$$

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_z}{\partial x} = \alpha_2 \quad \Rightarrow \quad v_x = \alpha_2 z \quad ; \quad v_z = \alpha_2 x$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} = \alpha_3 \quad \Rightarrow \quad v_y = \alpha_3 x \quad ; \quad v_x = \alpha_3 y$$

$$\Rightarrow \quad v_x = \overset{\overset{\alpha_3}{\downarrow}}{z}\,\overset{\overset{\alpha_2}{\downarrow}}{y} \quad ; \quad v_y = \overset{\overset{\alpha_1}{\downarrow}}{x}\,\overset{\overset{\alpha_3}{\downarrow}}{z} \quad ; \quad v_z = \overset{\overset{\alpha_1}{\downarrow}}{x}\,\overset{\overset{\alpha_2}{\downarrow}}{y}$$

$$\Rightarrow \quad \bar{v} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$

$$\bar{\nabla}\cdot\bar{v} = 0$$

Plot the above vector field and show its variation.

Product Rule of $\bar{\nabla}$

Gradient:

$$\bar{\nabla}(fg) = f\bar{\nabla}g + g\bar{\nabla}f$$

$$\bar{\nabla}(\bar{A}.\bar{B}) = \bar{A}\times(\bar{\nabla}\times\bar{B}) + \bar{B}\times(\bar{\nabla}\times\bar{A}) + (A.\bar{\nabla})\bar{B} + (\bar{B}.\bar{\nabla})\bar{A}$$

Divergence

$$\nabla.(f\bar{A}) = \bar{\nabla}f.\bar{A} + f(\bar{\nabla}.\bar{A})$$

$$\nabla.(\bar{A}\times\bar{B}) = \bar{B}.(\bar{\nabla}\times\bar{A}) - \bar{A}.(\bar{\nabla}\times\bar{B})$$

Curl

$$\bar{\nabla}\times(f\bar{A}) = f(\bar{\nabla}\times\bar{A}) + \bar{\nabla}f\times\bar{A}$$

$$\bar{\nabla}\times(\bar{A}\times\bar{B}) = (\bar{B}.\bar{\nabla})\bar{A} - (\bar{A}.\bar{\nabla})\bar{B} + \bar{A}(\bar{\nabla}.\bar{B}) - \bar{B}(\bar{\nabla}.\bar{A})$$

2^{nd} Derivatives

$$\bar{\nabla} \cdot \bar{\nabla} f = \nabla^2 f \text{ (Scalar)}$$

$\rightarrow \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \partial^2 f / \partial z^2$

$\rightarrow$ Laplacian of f

$$\bar{\nabla} \times \bar{\nabla} f = 0$$ Always!! Prove it

$$\bar{\nabla} (\bar{\nabla} \cdot \bar{v}) = ?$$

$$\bar{\nabla} \cdot (\bar{\nabla} \times \bar{v}) = 0$$ Always!! Prove it

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{v}) = \bar{\nabla} (\bar{\nabla} \cdot \bar{v}) - \nabla^2 (\bar{v})$$

Theorem: If $\nabla \times$ of a vector field is zero i.e. $\bar{\nabla} \times \bar{A} = 0$
$\Rightarrow \vec{A}$ is given by the gradient of a scalar

i.e. $\boxed{\bar{\nabla} \times \bar{A} = 0 \Rightarrow \bar{A} = \bar{\nabla} \psi}$

Theorem: If $\bar{\nabla} \cdot$ of a vector field is zero i.e. $\bar{\nabla} \cdot \bar{A} = 0$
$\Rightarrow \bar{A}$ is given by the curl of a vector field

i.e. $\boxed{\bar{\nabla} \cdot \bar{A} = 0 \Rightarrow \bar{A} = \bar{\nabla} \times \bar{B}}$

Integral Calculus of Vector Fields

1. Line Integral $\int_a^b \bar{A} \cdot \bar{dl}$ = Sum of projection of $\bar{A}$ along $\bar{dl}$ at all points

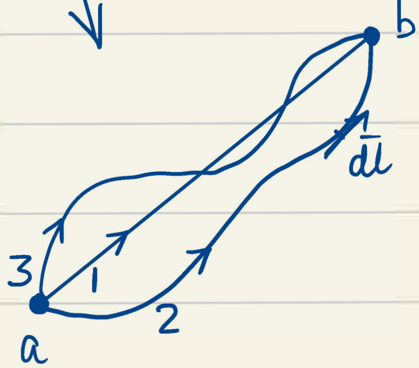
infinitsimal line vector

$$\bar{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

path

$$= d\rho\,\hat{\rho} + \rho\, d\theta\,\hat{\theta} + dz\,\hat{k}$$

$$= dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\varphi\,\hat{\varphi}$$



$\bar{dl}$ at any point is along the local tangent to the path.

If $a = b \Rightarrow$ path is closed $\oint \bar{A} \cdot \bar{dl}$

$\int \bar{dl} \neq 0$

Called a closed loop integral

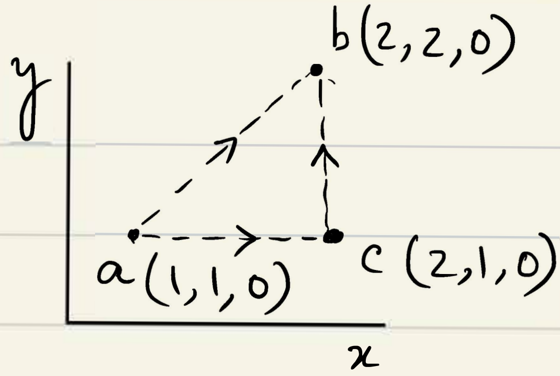
- Example: Work Done in a Force field

$$WD = \int_a^b \bar{F} \cdot \bar{dl}$$

In general the line integral of a vector field between points "a" and "b" is dependent on the path. But for a particular class of vector fields the integral becomes independent of the path and is dependent only on the end point values. Such a vector field is called a Conservative Vector field $\Rightarrow \oint \bar{A} \cdot \bar{dl} = 0$ and we can show

$$\boxed{\bar{A} = \bar{\nabla} \psi}$$

• Example



$$\bar{v} = y^2\,\hat{i} + 2x(y+1)\,\hat{j}$$

I $a \to b$ $\quad \displaystyle\int_a^b \bar{v}\cdot d\bar{l} = \int_a^b \left(y^2\hat{i} + 2x(y+1)\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$

$$= \int y^2\,dx + 2x(y+1)\,dy$$

path $a \to b$ $\Rightarrow$ Eqn of path $y - x = 0 \Rightarrow dx = dy$

$$\Rightarrow \int_a^b x^2\,dx + 2x(x+1)\,dx = \int_1^2 \left(x^2 + 2x^2 + 2x\right)dx = 10$$

II $a \to c + c \to b$: $\displaystyle\int_a^b \bar{v}\cdot d\bar{l} = \int_a^c \bar{v}\cdot d\bar{l} + \int_c^b \bar{v}\cdot d\bar{l}$

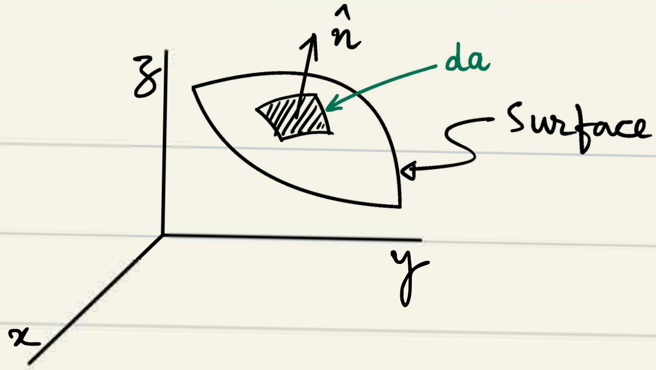
$$= \int_a^c \bar{v}\cdot d\bar{x} + \int_c^b \bar{v}\cdot d\bar{y}$$

$$= 11$$

$\Rightarrow$ For this vector field the line integral is path dependent.

and $\oint \bar{v}\cdot d\bar{l} \neq 0$

2. Surface Integrals:



$$\iint \bar{v} \cdot d\bar{a} = \iint \bar{v} \cdot \hat{n} \, da$$

$\hat{n}$: unit vector ⊥ to infinitsimal surface

→ Surface usually has 2 normals – choose any!

→ For a closed surface choose the outward normal

Say $\psi(x, y, z)$ or $\psi(r, \theta, \phi)$ is the Equation of a surface then $\bar{\nabla}\psi$ points along the normal!

③ Volume Integrals:

$$\iiint \bar{v} \, d\tau \quad \text{or} \quad \iiint f \, d\tau$$

$d\tau$: infinitsimal volume element

Integral Theorems :-

1. $\psi(x,y,z)$ ← any scalar function

$$\int_a^b d\psi = \psi(b) - \psi(a) \qquad \text{Now } d\psi = \bar{\nabla}\psi \cdot d\bar{l}$$

$$\Rightarrow \int_a^b \bar{\nabla}\psi \cdot dl = \psi(b) - \psi(a) \Rightarrow \oint \bar{\nabla}\psi \cdot d\bar{l} = 0$$

Line integrals of gradients are path independent

$$\Rightarrow \boxed{\text{if } \oint \bar{A} \cdot d\bar{l} = 0 \Rightarrow \bar{A} = \bar{\nabla}\psi \Rightarrow \bar{\nabla} \times \bar{A} = \bar{\nabla} \times (\nabla\psi) = 0}$$

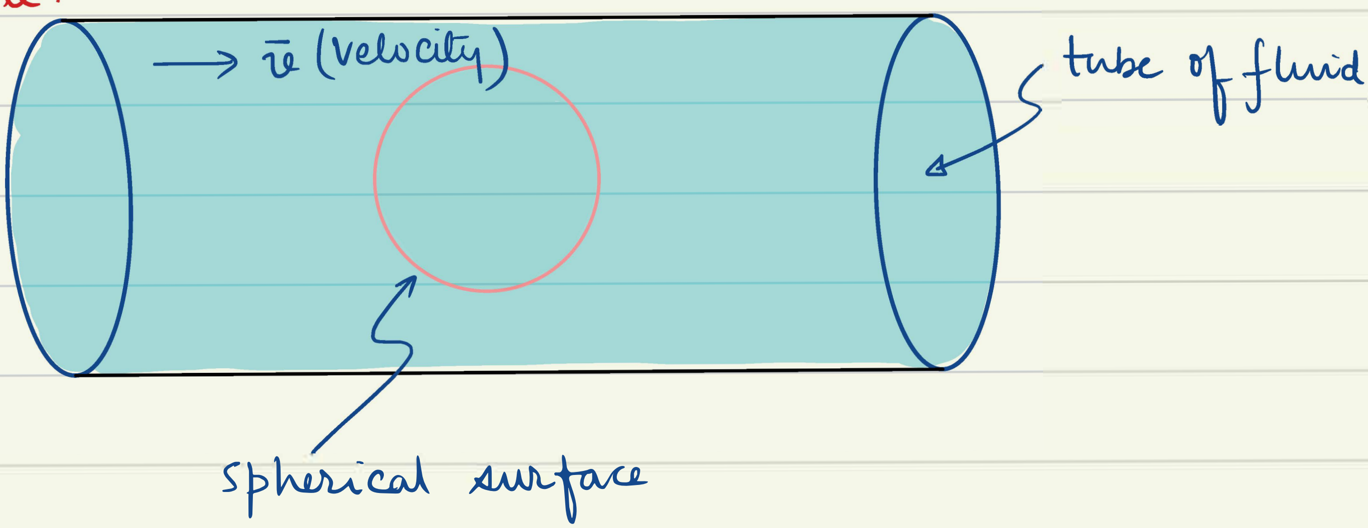
2. $\oint \bar{A} \cdot d\bar{a} = \int_{vol} (\bar{\nabla} \cdot \bar{A})\, d\tau$: DIVERGENCE THEOREM

Gauss' Law or Green's Theorem

Surface Integral of a vector field over a closed Surface is equal to the volume integral of $\bar{\nabla}\cdot$ of the field integrated over the enclosed volume.

$\oint \bar{A} \cdot d\bar{a} \rightarrow$ denotes net flux of the field through the surface

- Example:



LHS: $\oint \bar{v} \cdot d\bar{a} = ?$

Eqn of surface $x^2 + y^2 + z^2 = R^2$

$$\psi(x, y, z) = 0$$

$$\bar{\nabla}\psi = 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\hat{n} = \frac{\bar{\nabla}\psi}{|\nabla\psi|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{1/2}}$$

unit vector ⊥'r to surface

$\bar{v} = \alpha \hat{i}$ (say)

$$\Rightarrow \oint \bar{v} \cdot d\bar{a} = \oint \bar{v} \cdot \hat{n}\, da = \iint \frac{\alpha x}{(x^2+y^2+z^2)^{1/2}} R^2 \sin\theta \, d\theta \, d\varphi$$

$$= \iint \frac{\alpha \cancel{R} \sin\theta \cos\varphi}{\cancel{R}} R^2 \sin\theta \, d\theta \, d\varphi$$

$$= R^2 \alpha \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \cos\varphi \, d\varphi = 0$$

Total flux across the closed surface $= 0$

$\Rightarrow$ Amount of fluid entering the sphere per unit time
$=$ amount of fluid leaving per unit time.

RHS: $\int_{Vol} (\bar{\nabla} . \bar{v}) \, d\tau = 0 \quad \Rightarrow \because \bar{\nabla}.\bar{v}$ is a measure of source or sink

$\int \bar{\nabla}.\bar{v} \, d\tau = 0 \Rightarrow$ absence of source or sink inside the volume enclosed by surface.

If there are a lot of fluid "sources" inside the sphere $\Rightarrow$ rate of fluid creation $=$ rate of fluid leaving the volume i.e. flux across the surface

$\Rightarrow$ Measure of amount being created:

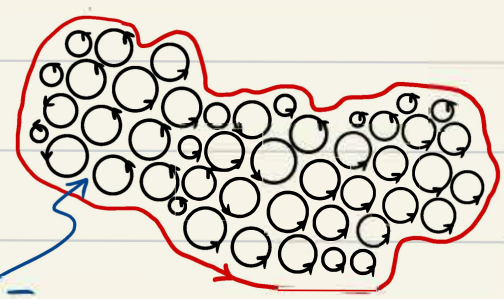
$\left.\begin{array}{l} \text{(1) } \Sigma \text{ sources} \\ \text{(2) } \Sigma \text{ leaving the surface} \end{array}\right\}$ They are Equal

- Example: $\bar{v} = \hat{r}/r^2$ is $\int_{Vol} (\nabla . \bar{v}) \, d\tau = \oint \bar{v} . \bar{da}$??

③ $\int_S (\bar{\nabla} \times \bar{v}) \cdot d\bar{a} = \oint \bar{v} \cdot d\bar{l}$: STOKES THEOREM

The $\bar{\nabla}\times$ measures the twist or degree of curl of $\bar{v}$ at a point.

⇒ The sum of all curls across a surface can be computed by calculating the line integral of the vector field along the bounding line.



Note: a closed loop can support ∞ surfaces → above is true for all surfaces

$\bar{\nabla} \times \bar{v}$

→ But if the surface is closed $\Rightarrow \oint d\bar{l} = 0 \Rightarrow \int_S \bar{\nabla} \times \bar{v} \cdot d\bar{a} = 0$

→ Surface integral is indep of surface chosen.

→ Ambiguity in direction - both are fine (be consistent)

Corollaries:

1. $$\int_{vol} \nabla\psi \, d\tau = \oint_S \psi \, d\bar{a}$$

2. $$\int_V (\nabla \times \bar{A}) \, d\tau = -\oint_S \bar{A} \times d\bar{a}$$

3. $$\int_V \left(\psi \nabla^2 \chi + \bar{\nabla}\psi . \nabla\chi\right) d\tau = \oint_S (\psi \bar{\nabla}\chi) . d\bar{a}$$

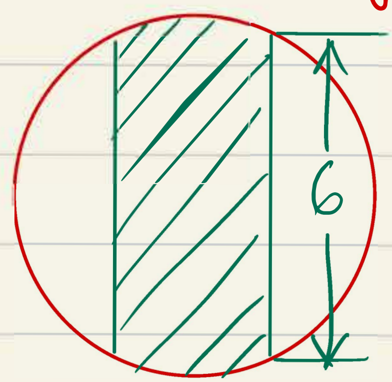
4. $$\int_V \left(\psi \nabla^2 \chi - \chi \nabla^2 \psi\right) d\tau = \oint (\psi \bar{\nabla}\chi - \chi \bar{\nabla}\psi) . d\bar{a}$$

5. $$\int_S \bar{\nabla}\psi \times d\bar{a} = -\oint \psi \, d\bar{l}$$

Homework: Prove all of the above starting from the 3 Integral Theorems.

Homework:

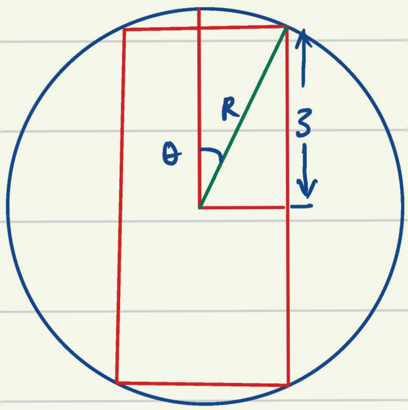
Take a sphere and scoop out a cylinder from the centre of siderial height 6.
What is the volume of the remaining part of the sphere?



Donot look at the solution on the next page but give an honest try. It is not difficult.

Solution :-

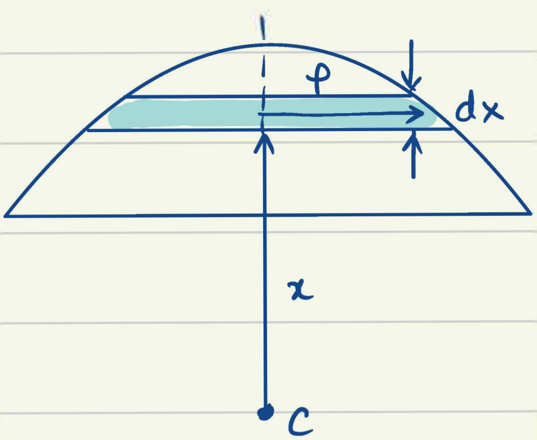
Volume of part removed $= V_{CYLINDER} + V_{CAP}$



$$= \pi R^2 H + V_{CAP}$$

$$= \pi (R^2 - 9)\, 6 + V_{CAP}$$

$$= 6\pi R^2 - 54\pi + V_{CAP}$$



$$V_{CAP} = \int_{vol} \pi \rho^2 \, dx$$

Now $x^2 + \rho^2 = R^2$

$$\Rightarrow \quad V_{CAP} = \int_3^R \pi \left(R^2 - x^2\right) dx$$

$$= \pi R^2 x \Big|_3^R - \frac{\pi x^3}{3} \Big|_3^R = \pi R^2 (R-3) - \frac{\pi}{3}\left(R^3 - 27\right)$$

$$= \pi R^3 - 3\pi R^2 - \frac{\pi R^3}{3} + \frac{27\pi}{3}$$

$$= \frac{2}{3}\pi R^3 - 3\pi R^2 + 9\pi$$

$$V_{CYL} + V_{CAP} = 6\pi R^2 - 54\pi + \left(\frac{2}{3}\pi R^3 - 3\pi R^2 + 9\pi\right) \times 2$$

$$= 6\pi R^2 - 54\pi + \frac{4}{3}\pi R^3 - 6\pi R^2 + 18\pi$$

$$\text{Volume left} = V_{sphere} - \left(V_{CYL} + V_{CAP}\right)$$

$$= \cancel{\frac{4}{3}\pi R^3} - \cancel{6\pi R^2} + 54\pi - \cancel{\frac{4}{3}\pi R^3} + \cancel{6\pi R^2} - 18\pi$$

$$= 36\pi$$

The answer is independent of the radius of the sphere!

Obviously $R > 3$

Remember!

scalar potential

① if $\bar{\nabla} \times \bar{A} = 0 \Rightarrow \bar{A} = \bar{\nabla}\psi$ ψ is not unique

$\psi \to \psi + \text{const.}$

$\Rightarrow \oint \bar{A} \cdot \bar{dl} = 0$

$\bar{A}$ is a Conservative vector field

vector potential

② If $\bar{\nabla} \cdot \bar{A} = 0 \Rightarrow \bar{A} = \bar{\nabla} \times \bar{B}$ $\bar{B}$ not unique

$\bar{B} \to \bar{B} + \bar{\nabla}\chi$

$\Rightarrow \oint \bar{A} \cdot \bar{da} = 0$

$\downarrow$

$\nabla \times (\bar{B} + \bar{\nabla}\chi)$

$= \bar{\nabla} \times \bar{B} + \cancelto{0}{\bar{\nabla} \times (\bar{\nabla}\chi)}$

$= \bar{\nabla} \times \bar{B}$

$\bar{v} = \hat{r}/r^2$

Divergence Theorem

is $\int_{vol} (\nabla \cdot \bar{v})\, d\tau = \oint \bar{v} \cdot d\bar{a}$??

$$\bar{\nabla} \cdot \bar{v} = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)}_{\text{divergence operator in Sp polar co-ord}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = 0$$

divergence operator
in Sp polar co-ord

$$\Rightarrow \int_{vol} (\bar{\nabla} \cdot \bar{v})\, d\tau = 0$$

infinitesimal surface area with
r = constant

$$\oint_S \bar{v} \cdot d\bar{a} = \oint_S \frac{\hat{r}}{\cancel{r^2}} \cdot \cancel{r^2} \sin\theta\, d\theta\, d\varphi\, \hat{r} = 4\pi \neq 0$$

$$\Rightarrow \int \nabla \cdot \bar{v}\, d\tau \neq \oint \bar{v} \cdot d\bar{a} \quad \text{for } \bar{v} = \hat{r}/r^2$$

Problem: $\bar{\nabla} \cdot \bar{v} = 0$ everywhere but at $r = 0$

What happens at $r=0$??

and the volume integral of $\bar{\nabla} \cdot \bar{v}$ including $r=0$

is $= 4\pi$ ← How do we get that ?

invented by Paul Dirac

The Dirac Delta Function

In 1 dimension

$$\delta(x) = 0 \quad \text{at } x \neq 0$$
$$= \infty \quad \text{at } x = 0$$

Function is zero everywhere except for an infinite spike at the origin

and $$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Actually its not a proper function at all

$$\delta(x) = \lim_{a \to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$

Properties:

① $$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

② $$\delta(\alpha x) = \delta(x)/\alpha$$

③ $$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

⑤ $\theta(x)$: Heaveside Step Fn

$$\theta'(x) = \delta(x)$$

④ $$\delta(-x) = \delta(x)$$

⑥ $$\frac{d}{dx}\delta(x) = -\frac{\delta(x)}{x}$$

δ Function in 3D

$$\delta^3(\bar{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

and $\displaystyle\int_{\text{all space}} \delta^3(\bar{r})\, d\tau = 1$

$$\because \int \left(\bar{\nabla} \cdot \frac{\hat{r}}{r^2}\right) d\tau = 4\pi$$

$$\Rightarrow \bar{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\, \delta^3(\bar{r})$$

$$\Rightarrow \int 4\pi\, \delta^3(r)\, d\tau = 4\pi$$

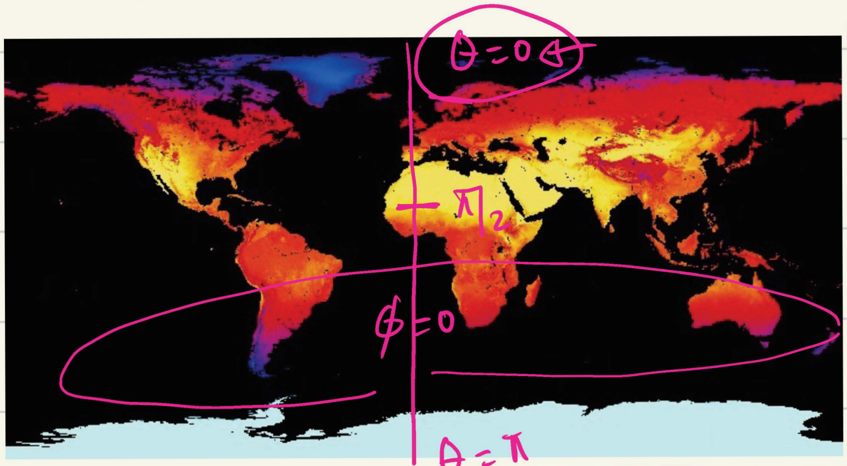
Note: $\delta^3(\bar{r}) = \infty$ at $r \to 0$

$= 0$ for $r \neq 0$

We'll see more of δ later.

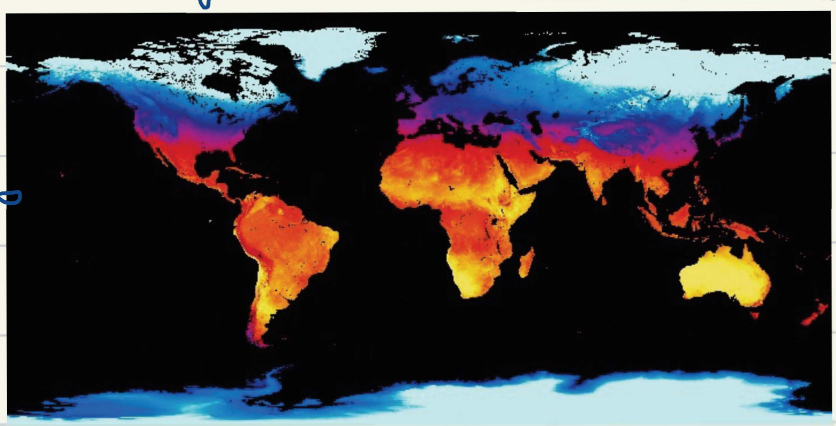
Average Land Surface Temperature

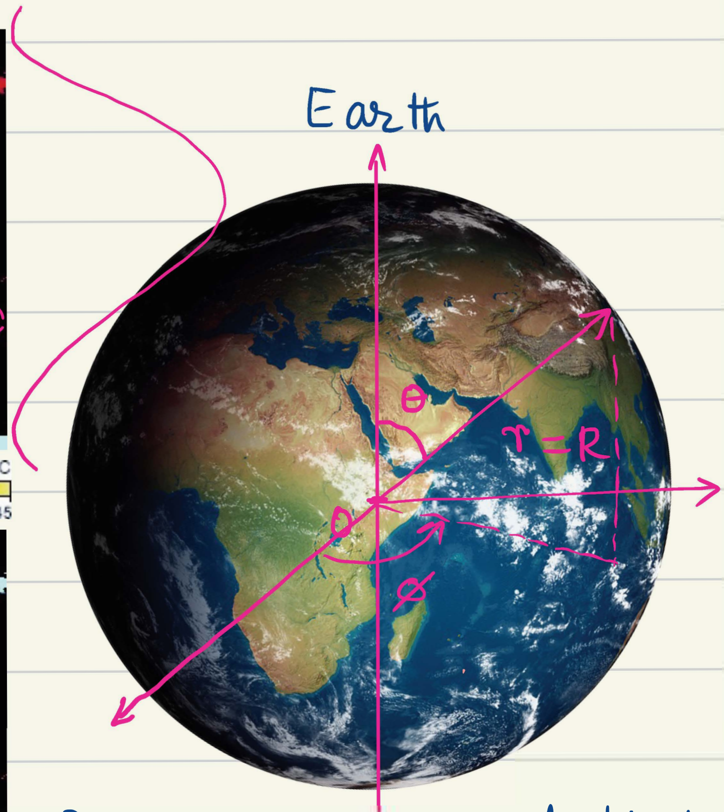
June 2001



nasa.gov

January 2001





Develop an analytical expression to model the temperature at any point on Earth.

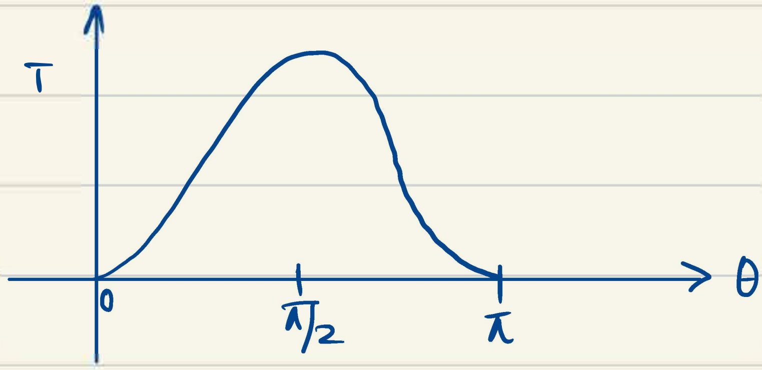
Owing to the spherical symmetry of the system we'll use the spherical polar co-ordinate system.

$$T(r, \theta, \varphi)$$

$r \rightarrow R-h_1 \rightarrow R+h_2$ $\quad \theta \rightarrow 0 - \pi$ $\quad \phi \rightarrow 0 \rightarrow 2\pi$

$h_1 / h_2 \ll R$

T would be lowest at the poles and hottest at the Equator



$\Rightarrow$ $T = T_1 \sin^2\theta$ $\leftarrow$ if you donot want $T(\theta=0)=0$ then add an offset.

So how does temp. vary with height at the surface of the Earth?

$\perp$r to the Earth's surface T varies as $T(r) = T_0\, e^{-\alpha(r-R)}$

T_0 and α are constants, R: radius of the earth

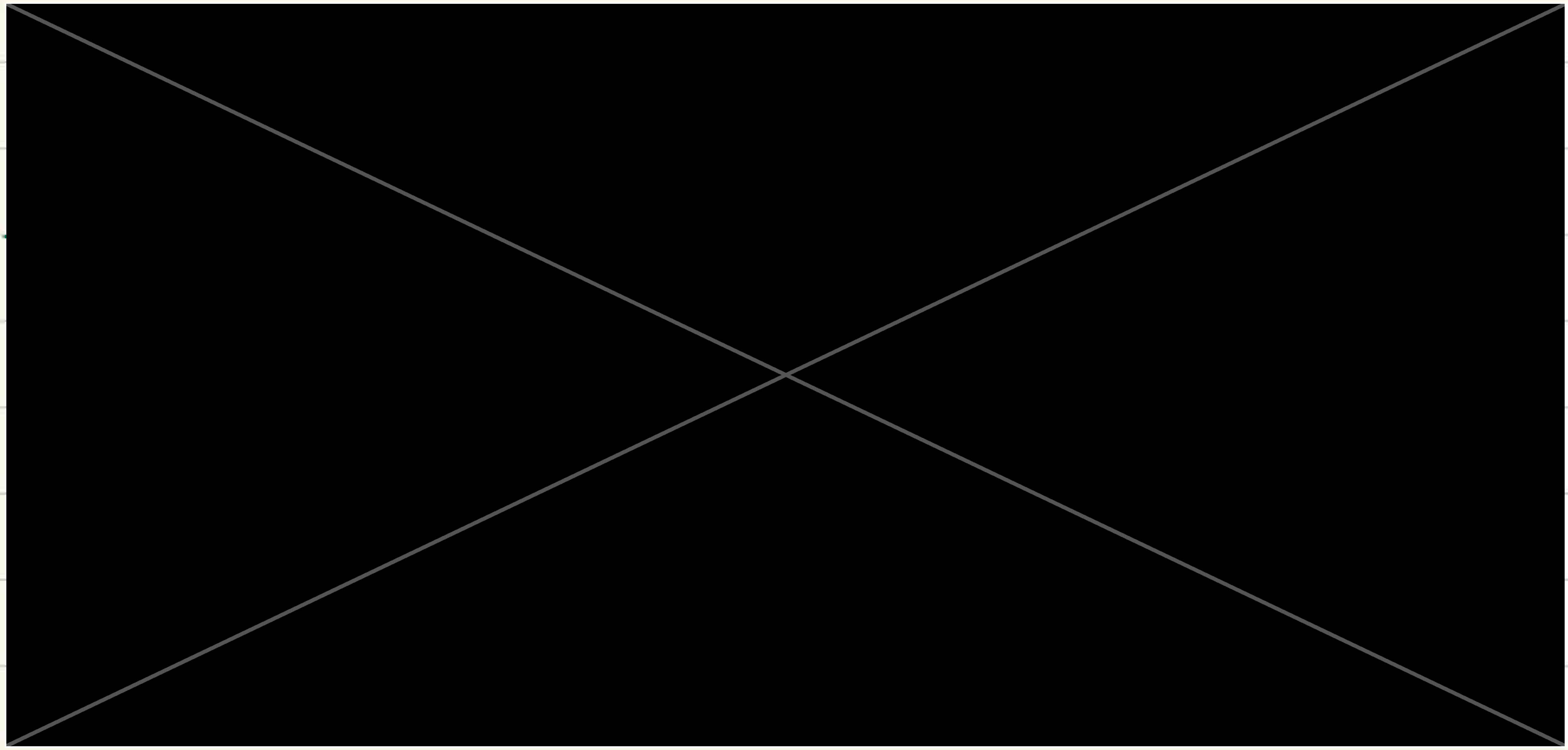
The above information is adequate to propose an analytic model that gives the daily average.

$$\boxed{T(r,\theta) = \left\{T_0 + T_1 \sin^2\theta\right\} e^{-\alpha(r-R)}}$$

Finally if we want variation within a day, the point facing the sun would be the hottest at any time of the day.

Thus T would vary with ϕ.

How do you think that should work?



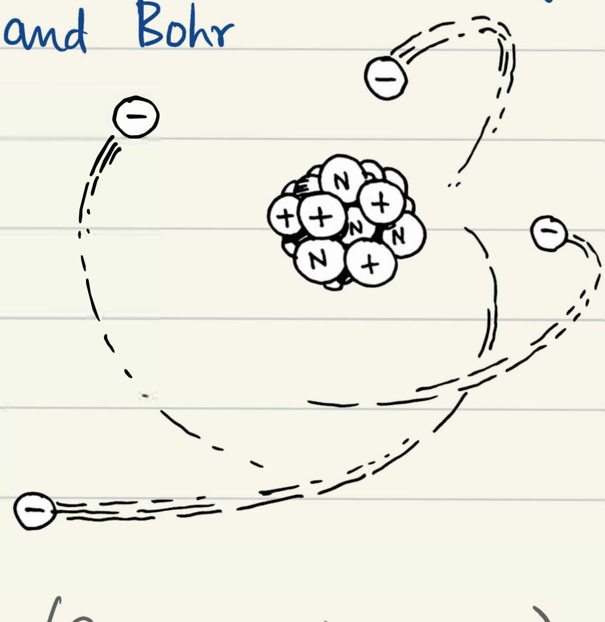
Chapter 2 Electrostatics

Electrostatics : All charges at rest / stationary
No motion / dynamics

BENJAMIN FRANKLIN (1706-1790) NAMED THE TWO KINDS OF CHARGES **POSITIVE** AND **NEGATIVE**.

and charging was due to the excess or lack of one of them. Franklin called that "mobile" charge +ve !

→ Modern Concept of Atom + Charges
Rutherford and Bohr



e^- : -ve -1.6×10^{-19} C

p : +ve $\underbrace{1.6 \times 10^{-19} \text{ C}}$

fundamental Quantum of Charge.

Cartoons from: The Cartoon Guide to Physics (Gonick & Huffman)

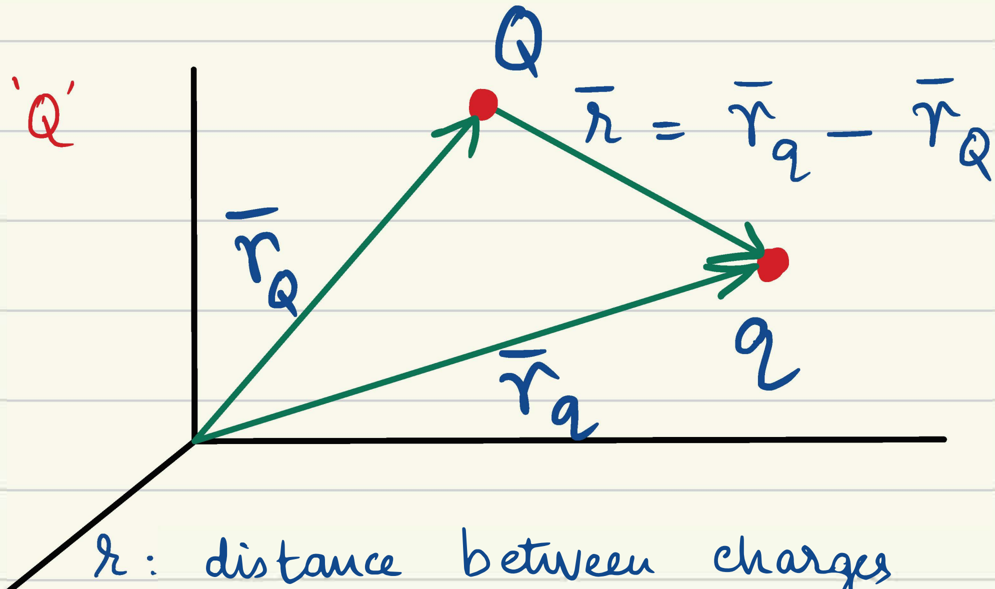
COULOMB (1736-1806)

1. Coulomb's Law ← Emperical Law

$$\vec{F}_q = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}$$

2 charges 'q' & 'Q'

force on a charge q due to a charge Q



r : distance between charges

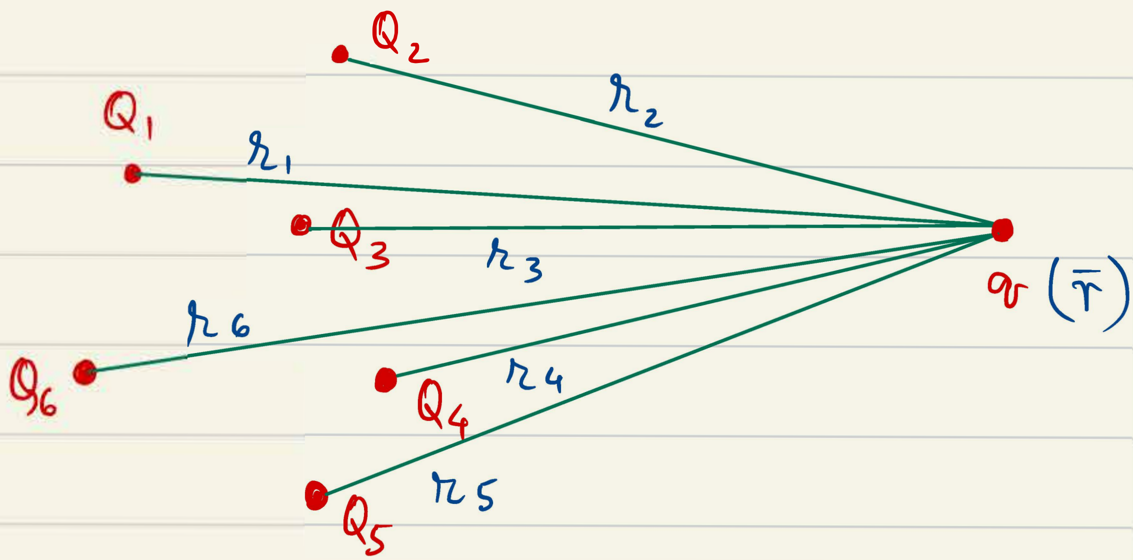
$\hat{r}$: unit vector along line joing charges

ϵ_0 : permittivity of free space

What if we had other charges? What is the total force on the test charge q ?

$$\bar{F}_q = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \dots = \frac{1}{4\pi\epsilon_0}\left\{ \frac{qQ_1}{r_1^2}\hat{r}_1 + \frac{qQ_2}{r_2^2}\hat{r}_2 + \frac{qQ_3}{r_3^2}\hat{r}_3 + \dots \right\}$$

Principle of Superposition!



$\bar{r}_i = \bar{r}_q - \bar{r}_{Qi}$: Separation of 'q' from each Q_i

$$\vec{F}_q = q \cdot \underbrace{\frac{1}{4\pi\epsilon_0}\left\{ \frac{Q_1\hat{r}_1}{r_1^2} + \frac{Q_2\hat{r}_2}{r_2^2} + \dots \right\}}_{\text{Net field at "q"!}}$$

$$\boxed{\vec{F}_q = q \cdot \vec{E}}$$

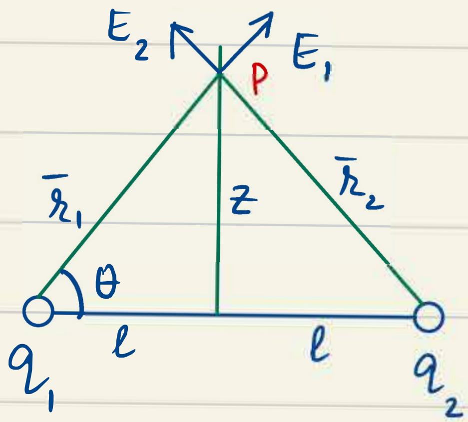
Force per unit charge

$\Rightarrow$ unit of $\bar{E}$ is N/C.

$\vec{E}$ field at a point is the Force experienced by a unit positive test charge at that point.

Discrete Charge Distributions

Ex 1 :



$$\bar{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1}{z^2 + l^2} \hat{r}_1$$

$$\bar{E}_2 = \frac{1}{4\pi\epsilon_0} \frac{q_2}{z^2 + l^2} \hat{r}_2$$

Case I : $q_1 = q_2 = q$ $\Rightarrow$ Horizontal Component Cancels

$$\Rightarrow E_{TOTAL} = 2 \cdot \frac{1}{4\pi\epsilon_0} \frac{q}{z^2+l^2} \cdot \sin\theta$$ ← magnitude of vertical Component

$$= \frac{1}{2\pi\epsilon_0} \frac{qz}{(z^2+l^2)^{3/2}}$$

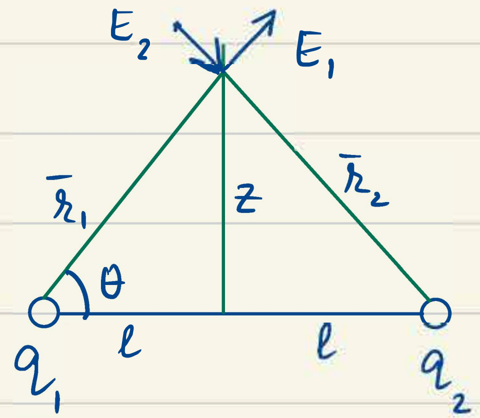
if $z \gg l$

$$E = \frac{2}{4\pi\epsilon_0} \frac{qz}{z^3(1+l^2/z^2)^{3/2}} \simeq \frac{q}{2\pi\epsilon_0 z^2}\left(1 - \frac{3}{2}\frac{l^2}{z^2}\right)$$

neglecting higher order terms.

$$E \simeq \frac{2q}{4\pi\epsilon_0 z^2} - \frac{3ql^2}{4\pi\epsilon_0 z^4}$$

Case II $\quad q_1 = -q_2 = q$ (equal and opposite charges)



$$\bar{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{q}{z^2+l^2} \hat{r}_1$$

$$E_2 = -\frac{1}{4\pi\epsilon_0} \frac{q}{z^2+l^2} \hat{r}_2$$

The vertical component cancels off

The Horizontal component points right

$$E_{TOTAL} = \frac{1}{4\pi\epsilon_0} \frac{2q}{(z^2+l^2)} \cos\theta = \frac{2q}{4\pi\epsilon_0} \frac{l}{(z^2+l^2)^{3/2}}$$

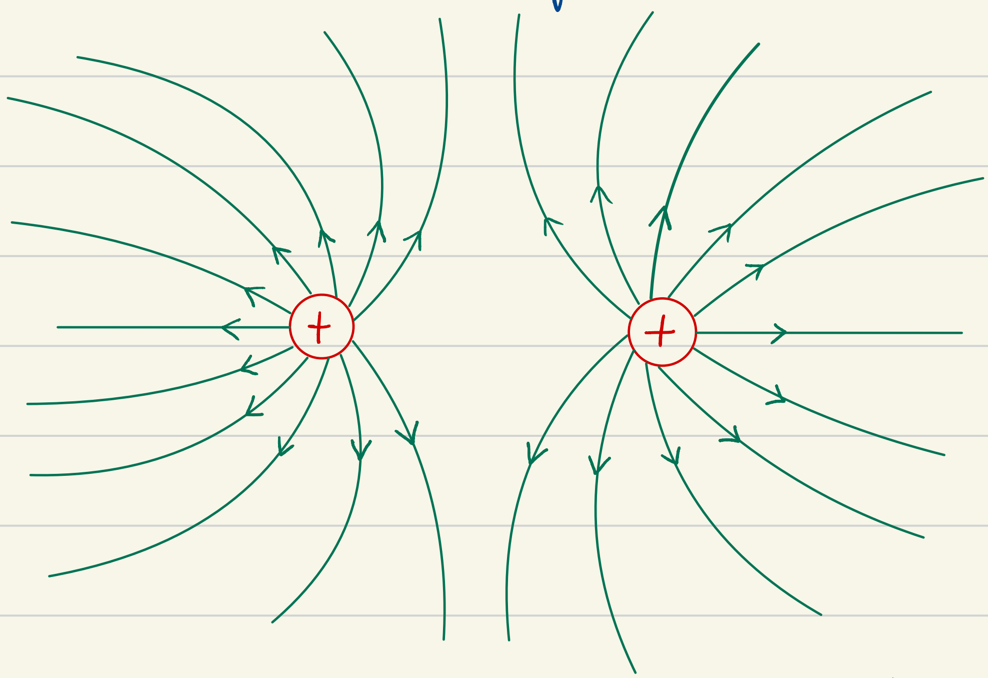
lt $z \to \infty$ $\Rightarrow$ $E \simeq \dfrac{2q}{4\pi\epsilon_0} \dfrac{l}{z^3}$

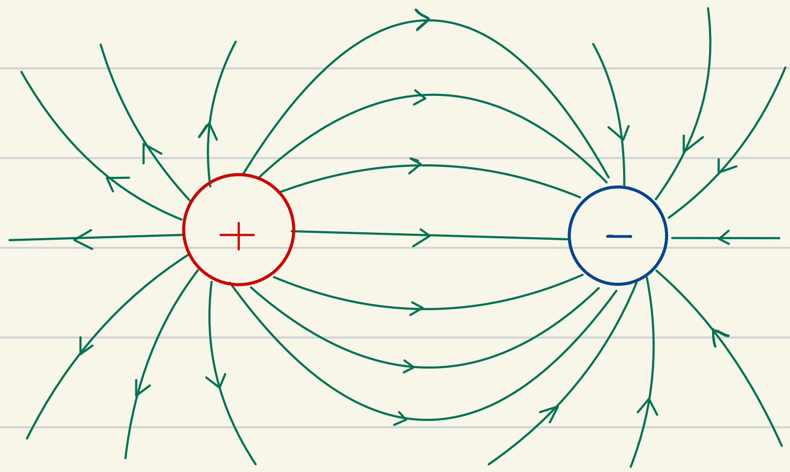
$l \ll z$

Note : The $\vec{E}$ field goes as $1/z^3$ and not $1/z^2$

i.e. $\vec{E}$ field of a dipole decays faster with distance.

How does the $\vec{E}$ field lines look in the 2 examples?





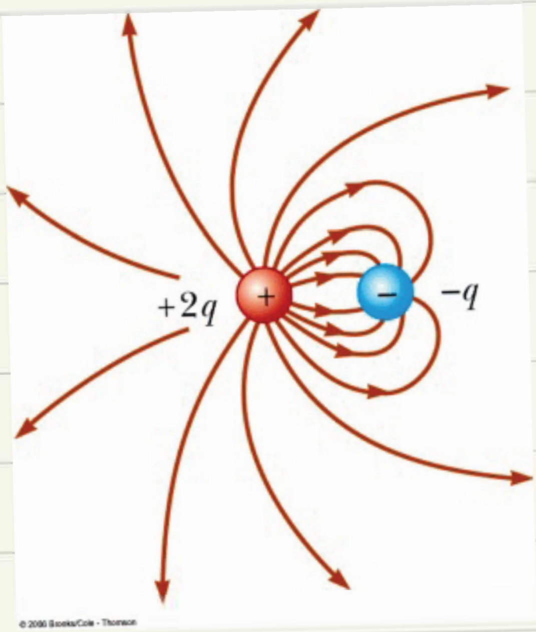
The field lines denote the direction of Force experienced by a unit positive test charge at that point.

Strength of the Force is denoted by the density of the Field lines.

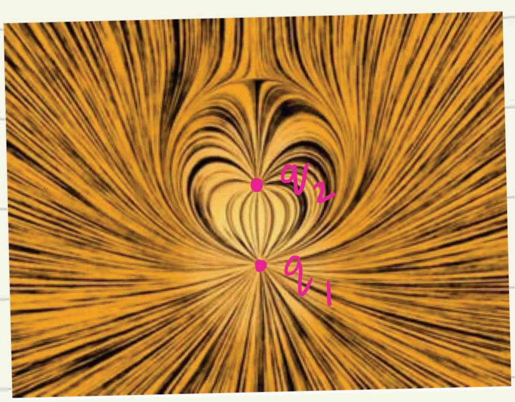
How do we calculate that density?

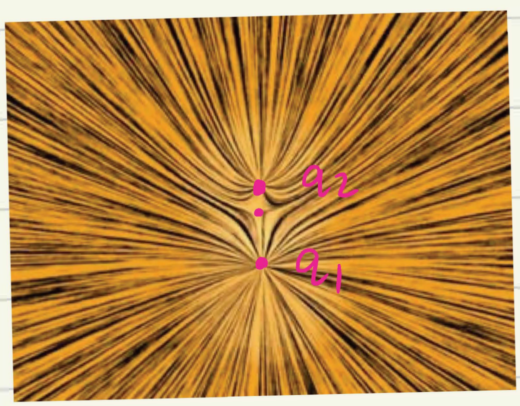
"PhET Interactive Simulations, University of Colorado Boulder, https://phet.colorado.edu."

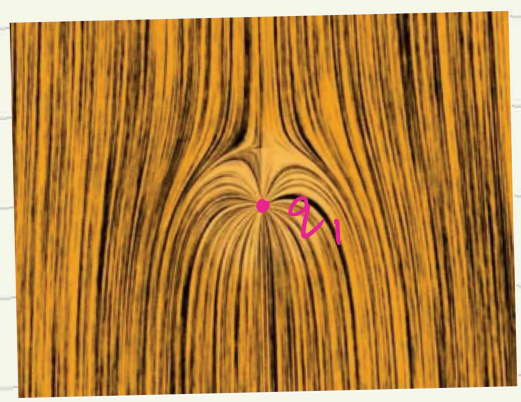
What happens if the -ve charge is ½ the +ve charge?



Can you guess the charge configurations in the following







→ q_1, q_2 are opp

$|q_1| > |q_2|$

q_1 + q_2 are of same sign

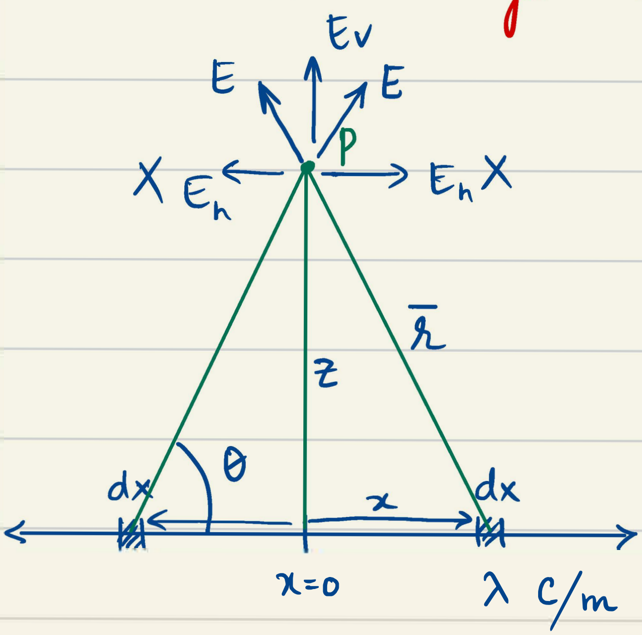
$|q_1| > |q_2|$

$q_1 > 0$

Background field magnitude

Calculating the Electric Field of Continuous charge distributions.

Ex 1: Line charge distribution



$$dQ = \lambda dx$$

Only the vertical component survives due to symmetry

$$dE = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{x^2+z^2} \sin\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda dx\, z}{(x^2+z^2)^{3/2}}$$

Total Field $\int dE = \frac{\lambda z}{4\pi\epsilon_0} \int \frac{dx}{(x^2+z^2)^{3/2}} \times 2$

For an infinite line charge $x = 0 \to \infty$

For a finite charge of Length L $x = 0 \to L/2$

Symmetrically placed charge.

Assume length is finite

$$E = \frac{2\lambda z}{4\pi\epsilon_0} \int_0^{L/2} \frac{dx}{(x^2+z^2)^{3/2}}$$

$$= \frac{2\lambda z}{4\pi\epsilon_0} \left. \frac{x}{z^2 (x^2+z^2)^{1/2}} \right|_0^{L/2}$$

$$= \frac{2\lambda \cancel{z}}{4\pi\epsilon_0} \frac{L/2}{z^{\cancel{2}} (L^2/4 + z^2)^{1/2}} = \frac{\lambda L}{4\pi\epsilon_0 \, z (z^2 + L^2/4)^{1/2}}$$

→ take the limit $L \ll z$ or $z \to \infty$ or $l \to 0$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{z(z^2 + L^2/4)^{1/2}} \longrightarrow \simeq \frac{1}{4\pi\epsilon_0} \frac{Q}{z^2}$$

Coulombs law

→ take the limit z is finite but $L \to \infty$

$$E = \frac{\lambda \cancel{L}}{4\pi\epsilon_0} \frac{1}{z \cancel{L} \left(\frac{1}{4} + z^2/L^2\right)^{1/2}}$$

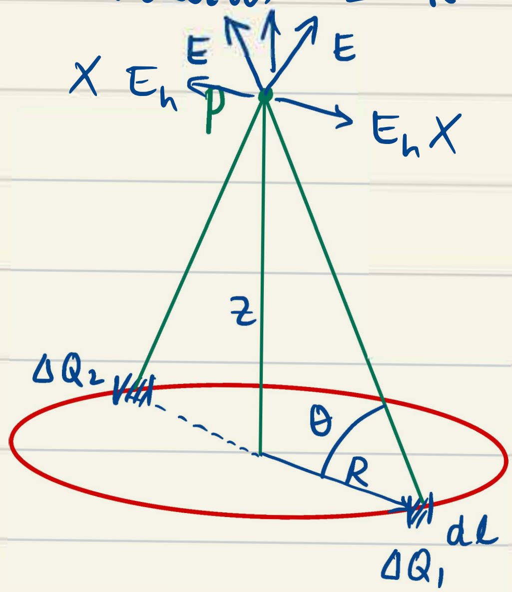
neglect that $\because z \ll L$

$$\Rightarrow \quad E = \frac{\lambda}{4\pi\epsilon_0 \, z (1/4)^{1/2}} = \frac{\lambda}{2\pi\epsilon_0 z}$$

$Q = 2\pi R \lambda$

Ex 2: Circular ring of charge density λ C/m

radius $= R$, Field point: distance z along the axis



→ problem has cylindrical sym.

$\Rightarrow (\rho, \theta, z)$

$\Rightarrow dl = R\, d\theta$

↑ infinitesimal line element

Horizontal components cancel off. and the vertical component survives ← symmetry

Vertical component of Field ≡ Net field

$$\int dE = \frac{1}{4\pi\epsilon_0} \int \frac{dQ\, z}{(z^2+R^2)^{3/2}}$$

$$= \frac{z}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} \int_0^{2\pi} R\, d\theta\, \lambda$$

$$\vec{E} = \frac{\lambda R z\, 2\pi}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} = \frac{Q z}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} \hat{z}$$

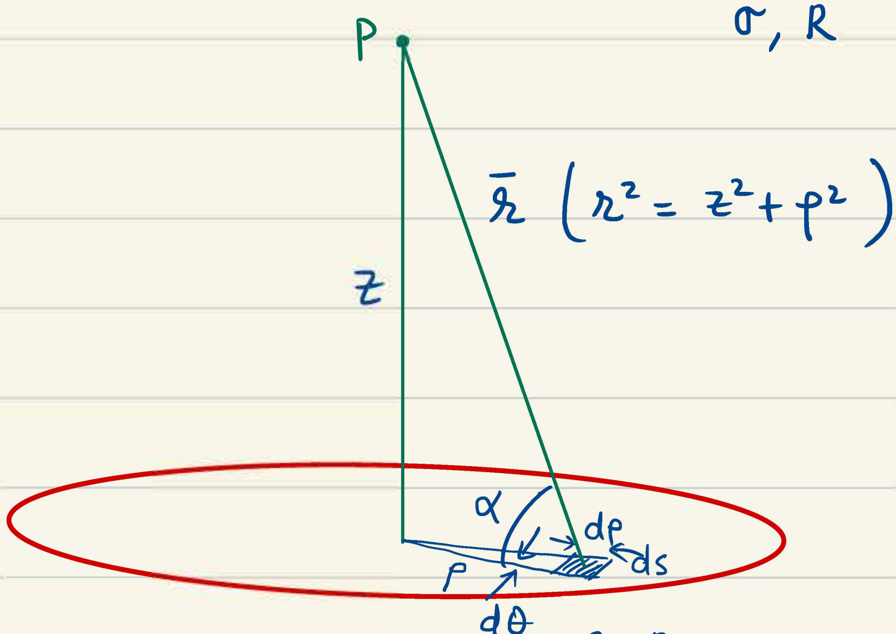
For $z \to \infty$ but R is finite $\Rightarrow R \ll z$

$$\vec{E} \simeq \frac{Q}{4\pi\epsilon_0 z^2} \hat{z}$$

Ex 3: Calculate the E field along the axis of a circular disk of radius R and surface charge density σ.

Total charge

$$\sigma, R \Rightarrow Q = \pi R^2 \sigma$$

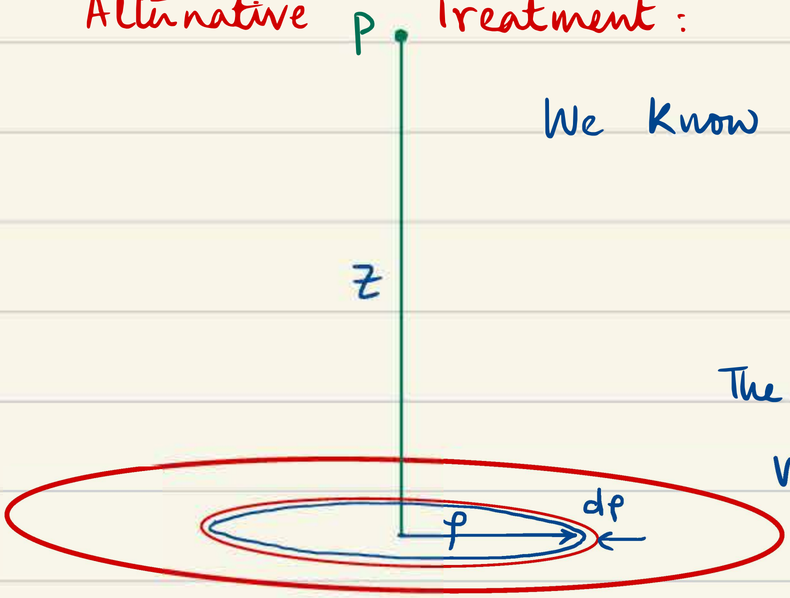


$$ds = \rho\, d\theta\, d\rho$$

$$\Rightarrow dQ = \sigma \rho\, d\theta\, d\rho$$

$$\uparrow \qquad \vec{E}_{\uparrow} = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi}\int_0^{R} \frac{\sigma \rho\, d\theta\, d\rho}{(z^2+\rho^2)^{3/2}} \cdot z = \ldots\ldots$$

Alternative Treatment:



We know the $\bar{E}$ field of a charged ring is

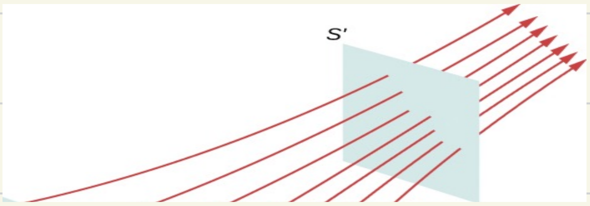
$$\frac{Qz}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} \hat{z}$$

The Field of this infinitsimal thick ring is

$$\int dE = \int_0^R \frac{2\pi\rho\, d\rho\, \sigma\, z}{4\pi\epsilon_0 (z^2+\rho^2)^{3/2}}$$

Field line Density

the number of field lines per unit area passing through a small cross-sectional area perpendicular to the electric field.



The number of field lines passing through both surfaces S and S' are the same. But $S > S'$

$\Rightarrow$ density is higher at S' than at S

$$\Rightarrow |E_{S'}| > |E_S|$$

Even though the density changes as a function of position

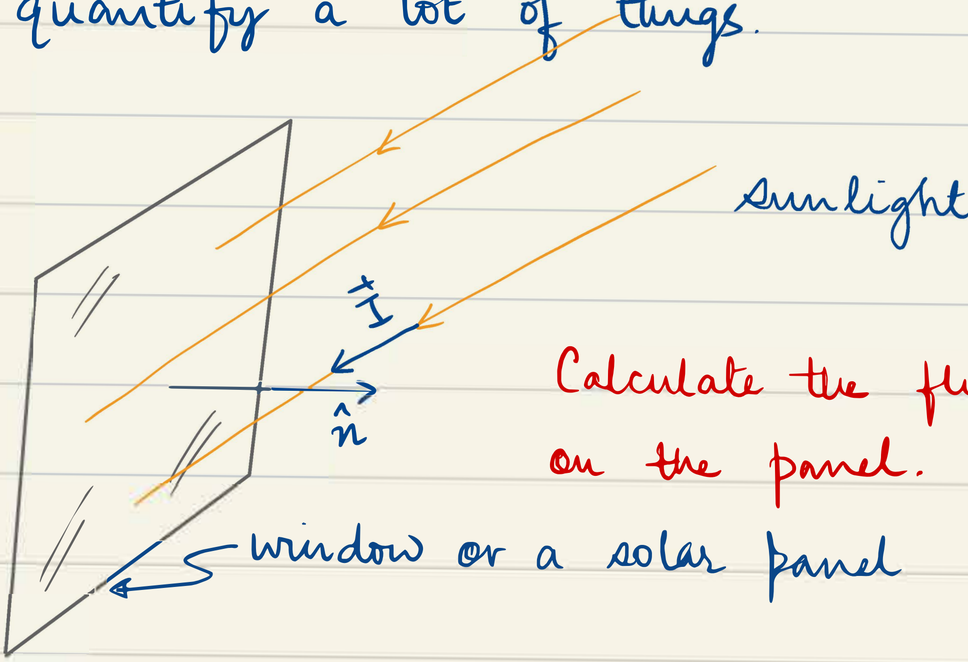
What about $\int_S \vec{E} \cdot \bar{da}$?

$$\int_S \vec{E} \cdot \bar{da} = \text{flux of the Electric field over a surface } S.$$

Electric Flux

The flux quantifies the amount of something that goes through a given area.

Flux is a very general concept that may be used to quantify a lot of things.



Calculate the flux of the sunlight on the panel.

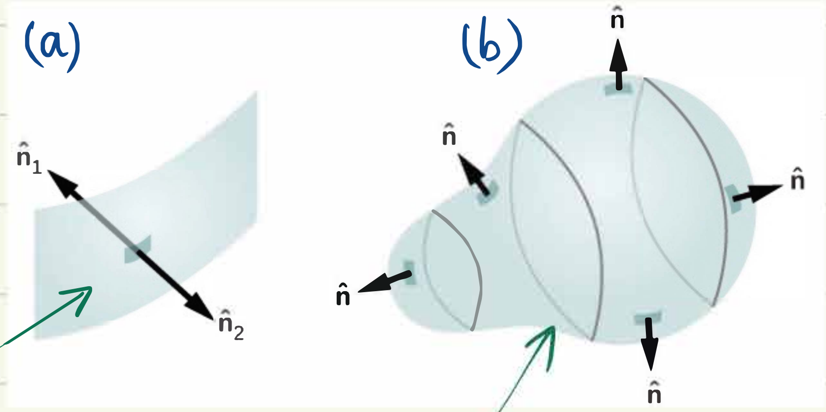
Integrate the normal component of the incident quantity over the entire area.

$$\text{Flux} = \sum_{\text{all area}} \bar{I} \cdot \hat{n} \longrightarrow \int_{\text{area}} \vec{I} \cdot \hat{n} \, da = \int \vec{I} \cdot \bar{da}$$

Remember:

1. For non-planar surfaces $\hat{n}$ is not unique but varies, i.e. position dependent
If you know the Eqn of the surface $\psi(x,y,z)=0$
$\bar{\nabla}\psi$ gives a vector $\perp$ to the surface at a point.

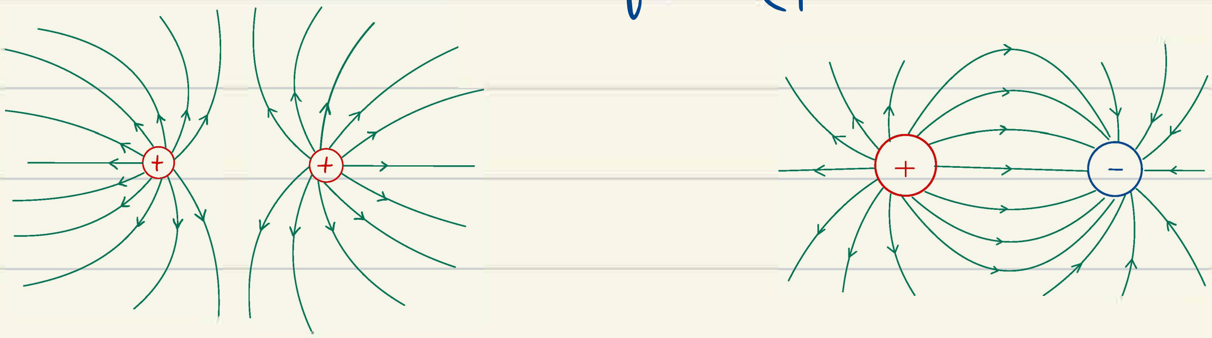
2. For an open surface there can be 2 unit vectors at any point
Choose any one –
But be consistent.



3. For a closed surface choose the outward normal at every point.

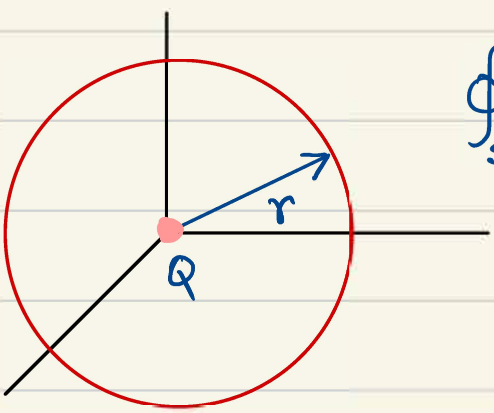
Gauss' Law.

Electric Flux ∝ <u>NUMBER</u> of $\vec{E}$ field lines passing through a surface. (pick out the ⊥'r component)



Field lines around any charge configuration varies from case to case.

Say we have a charge Q at the origin and Calculate its flux across a spherical surface of radius r.



$$\oint_S \vec{E}\cdot\bar{da} = \int \underbrace{\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}_{\vec{E}} \cdot \underbrace{r^2 \sin\theta\, d\theta\, d\varphi\, \hat{r}}_{\bar{da}}$$

$$= \frac{Q}{4\pi\epsilon_0}\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{Q}{4\pi\epsilon_0} \cdot 2 \cdot 2\pi = \frac{Q}{\epsilon_0}$$

> Note: the $1/r^2$ dependence of Coulomb's Law is vital here to remove "r" dependence

Thus the flux across this closed surface is a measure of the charge inside.

- What if the charge was off centred wrt the sphere?

Still the calculated Flux remains a constant!
But the contribution from various parts of the spherical surface changes.

- What if there are multiple charges inside?

Principle of Superposition: $\vec{E}(x,y,z) = \sum E_i$

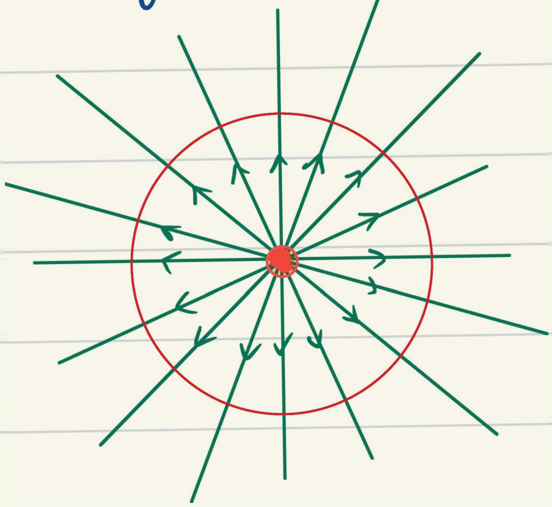
$$\Rightarrow \oint \vec{E} \cdot \bar{da} = \oint \sum_i \bar{E_i} \cdot \bar{da} = \sum_i \oint \bar{E_i} \cdot \bar{da} = \sum_i \frac{Q_i}{\epsilon_0}$$

$$= \text{Total charge enclosed} / \epsilon_0$$

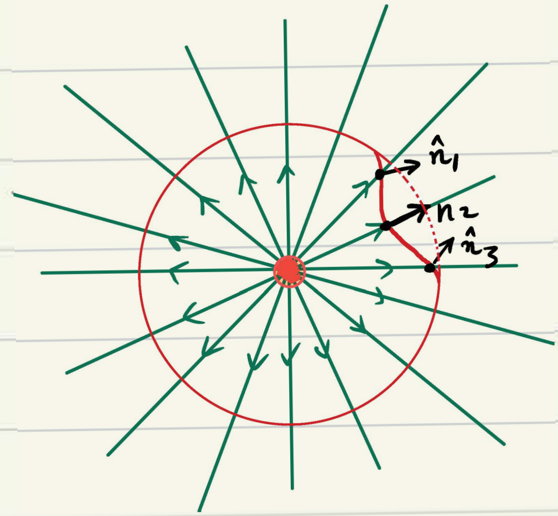
- What if the surface is not a sphere?

$\because$ the flux is a measure of number of field lines passing through it does not really matter if the surface is non-spherical.

Perfect Sphere



Dented Sphere



Contribution to the total flux is same from each point on the surface of the sphere.

At the 3 points the unit vectors are $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}_3$

Flux contribution from the 3 points.

$\bar{E}_1 \cdot \hat{n}_1 \, da_1$

$\bar{E}_2 \cdot \hat{n}_2 \, da_2$

$\bar{E}_3 \cdot \hat{n}_3 \, da_3 \qquad |E_1| = |E_3|$

"All" along the dented region

- Angle between $\vec{E}$ and $\hat{n}$: $\cos\theta < 1$
- the magnitude of the infinitesimal area projected ⊥'r to $\vec{E}$ are different! and $[da]_{\text{projected}} \propto r^2$
- $\vec{E}$ field strength are different $\propto 1/r^2$

$\Rightarrow$ Each point now contributes differently to the overall flux

But the sum of all the contributions is the same!

- What about contribution from charges outside a closed surface?

They have no net contribution to the total Flux.

$\vec{E} \cdot d\bar{a} \neq 0$ but $\oint \vec{E} \cdot d\bar{a} = 0$

Gauss' Law:

$$\boxed{\oint_S \vec{E} \cdot d\bar{a} = \frac{Q_{ENCLOSED}}{\epsilon_0}}$$ ← Integral form

$$\oint_S \vec{E} \cdot d\bar{a} = \int_V (\bar{\nabla} \cdot \bar{E})\, d\tau$$ ← Divergence Theorem

$$\Rightarrow \int_V \bar{\nabla} \cdot \bar{E}\, d\tau = \frac{Q_{ENCLOSED}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho\, d\tau$$

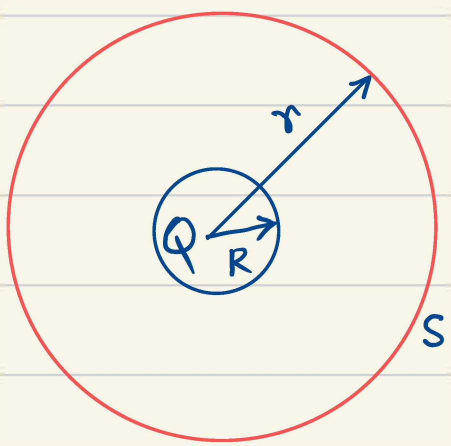
charge density at all points inside the surface S

$$\Rightarrow \boxed{\bar{\nabla} \cdot \bar{E} = \rho / \epsilon_0}$$ ← Differential Form

Remember: Gauss' Law is always true for any static charge distribution and any closed surface

But it is not always useful

Ex: Consider a sphere of radius R carrying a charge Q. Calculate the Field at a distance r from the centre of the sphere. $(r > R)$



$$\oint \vec{E} \cdot \bar{da} = Q/\epsilon_0$$

1. $\vec{E}$ // to $\bar{da}$ at all points } on the
2. $|\vec{E}|$ is same at all points } sphere

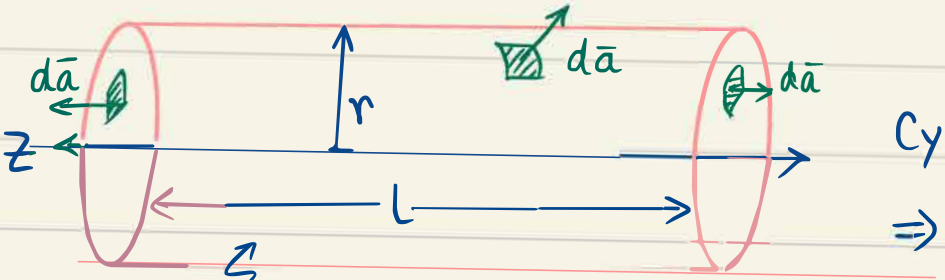
$$\Rightarrow \oint \vec{E} \cdot \bar{da} = \oint |\vec{E}| \, da = |\vec{E}| \int da = |\vec{E}| 4\pi r^2$$

Note: For any arbitrary surface points ① + ② would not be true ⇒ we cannot drag $|\vec{E}|$ outside the $\int$. Then Gauss' Law would not be useful!

$$\Rightarrow |\vec{E}| 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\Rightarrow |\vec{E}| = \frac{Q}{4\pi\epsilon_0 r^2} \Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

• Ex: Infinite Line charge λ C/m
Calculate $\bar{E}$ at a distance r from line.



Gaussian Surface : Closed surface

Cylindrical Symmetry
$\Rightarrow$ use cylindrical Co-ordinate System

The $\bar{E}$ field will point radially outward at all points i.e. along $\hat{\rho}$

$\Rightarrow$ ① $\vec{E}.\bar{da} = 0$ on the vertical end surfaces

② Curved Surface $\int \vec{E}.d\bar{a} = \iint |\vec{E}|\, r\, d\theta\, dz = \dfrac{Q_{enclosed}}{\epsilon_0}$

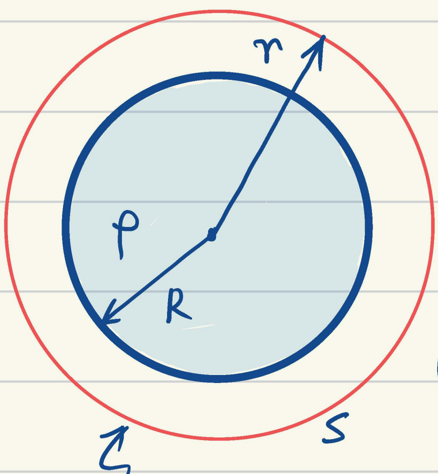
$$|\vec{E}|r \int_0^{2\pi} d\theta \int_0^{L} dz = \frac{\lambda L}{\epsilon_0} \Rightarrow |\bar{E}|\, 2\pi r L = \frac{\lambda L}{\epsilon_0}$$

$$\Rightarrow |\bar{E}| = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$\bar{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\rho}$$

- Calculate the $\bar{E}$ field of a spherical charge distribution of radius R and charge density ρ.

I: $r > R$



Gaussian Surface

① Spherical Symmetry

② Field outside points along $\hat{r}$

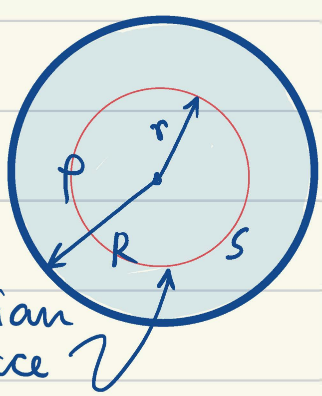
③ $$\oint_S \bar{E}.d\bar{a} = \frac{Q_{enclosed}}{\epsilon_0} = \frac{\frac{4}{3}\pi R^3 \rho}{\epsilon_0}$$

$$\Rightarrow |\vec{E}|\, 4\pi r^2 = \frac{4}{3}\frac{\pi R^3 \rho}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r}$$

II $r < R$

$$\oint_S \vec{E}.da = \frac{Q_{enclosed}}{\epsilon_0} = \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0}$$

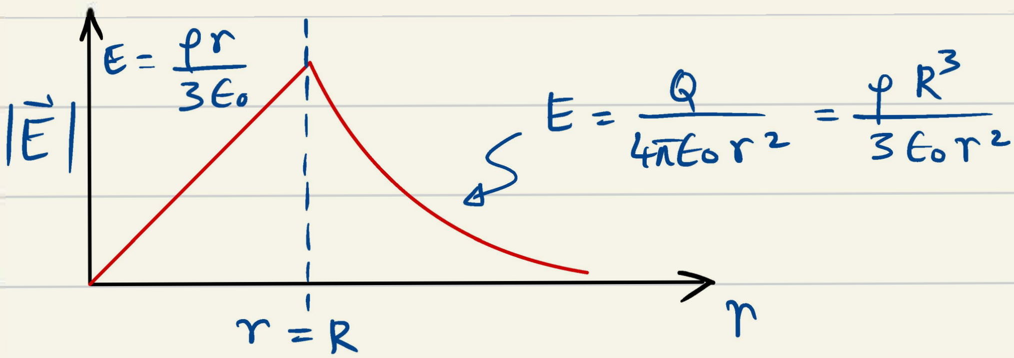


Gaussian Surface

$$\Rightarrow |\vec{E}|\, 4\pi r^2 = \frac{4}{3}\frac{\pi r^3 \rho}{\epsilon_0} \Rightarrow |\vec{E}| = \frac{\rho r}{3\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$$

Plot the $|\bar{E}|$ with r.



Visit the differential form of Gauss' Law.

$$\bar{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

For $r > R$ $\quad \bar{\nabla}\cdot\frac{\rho R^3}{3\epsilon_0 r^2}\hat{r} = 0 \Rightarrow$ no local charge density at any point $r > R$

For $r < R$ $\quad \bar{\nabla}\cdot\frac{\rho r}{3\epsilon_0}\hat{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\rho r}{3\epsilon_0}\right)$

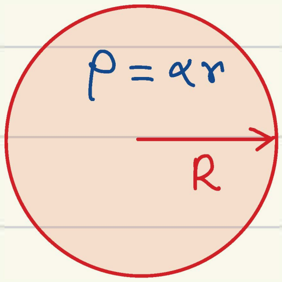
$$= \frac{1}{r^2}\cdot\frac{\rho}{3\epsilon_0}\frac{\partial}{\partial r}r^3 = \frac{\rho}{\cancel{3}\epsilon_0\cancel{r^2}}\cancel{3}\cancel{r^2}$$

$= \frac{\rho}{\epsilon_0} \Rightarrow$ local charge density at all points $r < R$ is ρ.

Homework

- Prove that the Field inside a uniformly charged shell is zero.

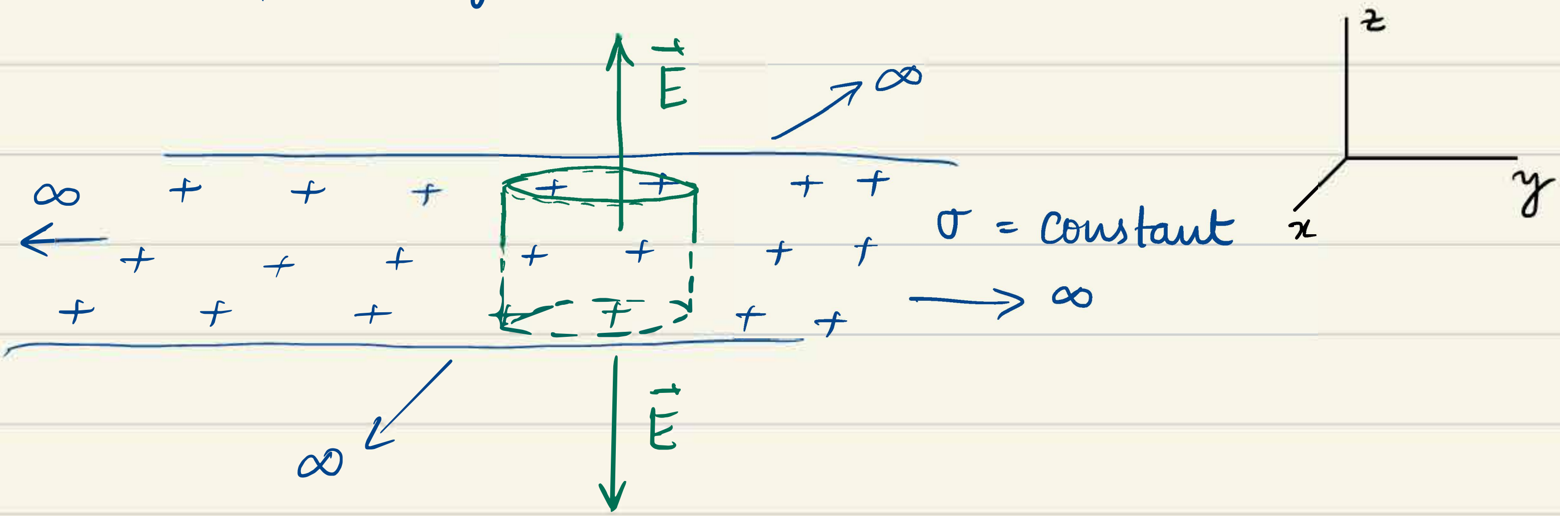
- Calculate the $\bar{E}$ if the volume charge density $\rho(r) = \alpha r$, inside a solid sphere of radius R.



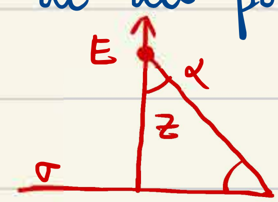
Ex: Field of an infinite sheet of charge.

Charge density σ.

Symmetry: linear $\Rightarrow$ Cartisian Co-ordinates



Electric field points $\perp$'r to the plane at all points.



From the Field of a finite disc:

$\vec{E}$ along the axis of the disc $\quad \vec{E} = \dfrac{\sigma}{2\epsilon_0}(1 - \cos\alpha)$

Take the lt $R \to \infty$ or $\alpha \to 90^\circ$

$$\Rightarrow \quad \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z}$$

Gaussian Surface: Cylinder straddling the ∞ plane

① $\vec{E}\cdot d\vec{a}$ on curved surface $= 0$

② $\displaystyle\int_{\text{top flat surface}} \vec{E}\cdot da = Q_{enc}/\epsilon_0 \quad \Longrightarrow \quad \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z} \quad z > 0$

$$= -\frac{\sigma}{2\epsilon_0}\hat{z} \quad z < 0$$

The curl of $\vec{E}$

A point charge: $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$

$$\vec{\nabla} \times \vec{E} = \frac{Q}{4\pi\epsilon_0} \left\{ \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \left(\frac{1}{r^2} \right) \right] \hat{\theta} - \frac{1}{r} \frac{\partial}{\partial\theta} \left(\frac{1}{r^2} \right) \hat{\varphi} \right\}$$

$$\bar{\nabla} \times \bar{E} = 0 \quad \Rightarrow \quad \vec{E} = \bar{\nabla}\psi$$

$\Rightarrow$ Stokes Theorem: $\int_S (\bar{\nabla} \times \bar{E}) \cdot d\bar{a} = \oint \vec{E} \cdot d\bar{l}$

$$\Rightarrow \oint \vec{E} \cdot d\bar{l} = 0$$

$\vec{E}$ is a conservative vector field

The corresponding scalar potential is $V(\bar{r})$

such that $\boxed{\vec{E}(\bar{r}) = -\bar{\nabla} V(\bar{r})}$

and $\int_a^b \vec{E} \cdot d\bar{l}$ is independent of the path from $a \to b$

The negative sign is a matter of Convention.

Electrostatic Potential

$$\vec{E}(\bar{r}) = -\bar{\nabla} V(\bar{r}) \longleftrightarrow \bar{\nabla}\times\bar{E} = 0$$

$$\int_a^b dV(\bar{r}) = V(b) - V(a) = \int_a^b \bar{\nabla} V(\bar{r}) \cdot \bar{dl} = -\int_a^b \bar{E}(\bar{r}) \cdot \bar{dl}$$

→ The information content of $\bar{E}(\bar{r})$ & $V(\bar{r})$ is the same.

→ 3 components of $\bar{E}(\bar{r})$ are not independent but are such that $\bar{\nabla}\times\bar{E} = 0$ always.

Any $V(\bar{r})$ is not unique but variable upto an additive constant.

Potential function $V(\bar{r})$ also follows the Principle of Superposition. $V(\bar{r}) = \sum_i V_i(\bar{r})$

Electrostatic Potential of a point charge:

$$V(\bar{r}) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential of Known Charge distributions / $\vec{E}$ fields

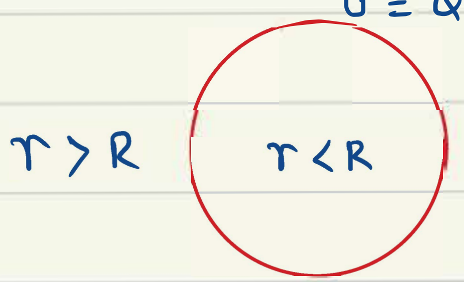
Known Charge distribution	Known $\vec{E}$ fields.
① Calculate for a point charge	$\vec{E}(\bar{r}) = -\nabla V(\bar{r}) \leftarrow$ Invert
② Use principle of Superposition	$V(\bar{r}) = -\int_{P}^{r} \vec{E}(\bar{r}').dl'$

In general it is far more easier + useful to obtain the $V(\bar{r})$ and then proceed to $\bar{E}(\bar{r})$.

P is a reference point where V is known!

Ex: Calculate the Electrostatic Potential of a charged shell.

$\sigma = Q/4\pi R^2$



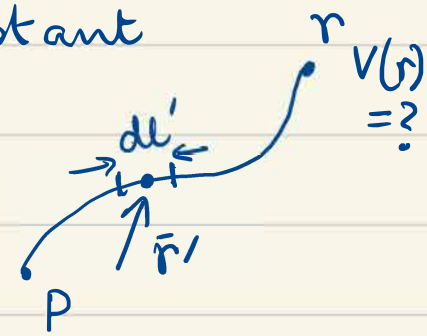
① Field $r \geqslant R$ $\quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$

② Field $r < R$ $\quad \vec{E} = 0$

$\vec{E} = 0$ for $r < R \Rightarrow -\nabla V(r) = 0$

$\Rightarrow V(r) = \text{Constant}$

$r \geq R \qquad V(\bar{r}) = -\int_P^r \bar{E}(\bar{r}') \cdot d\bar{l}'$



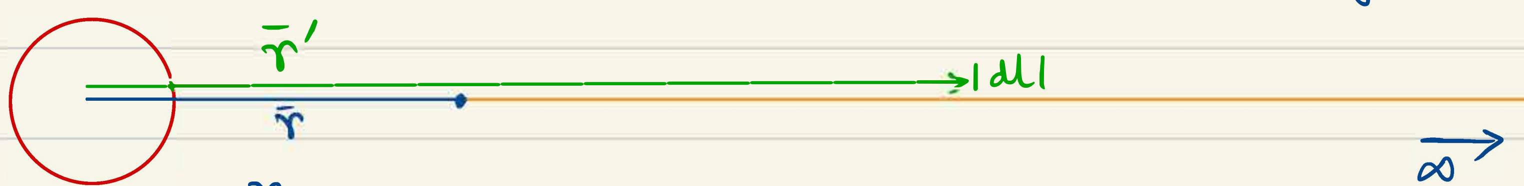
We choose the reference point P wrt which the potential at $\bar{r}$ is calculated as ∞

$$V(\bar{r} = \infty) = 0$$

$$\Rightarrow V(\bar{r}) = -\int_\infty^r \frac{1}{4\pi\epsilon_0} \frac{Q}{r'^2} \hat{r}' \cdot d\bar{r}'$$

$d\bar{l} = d\bar{r}$

unit vector along $\bar{r}$

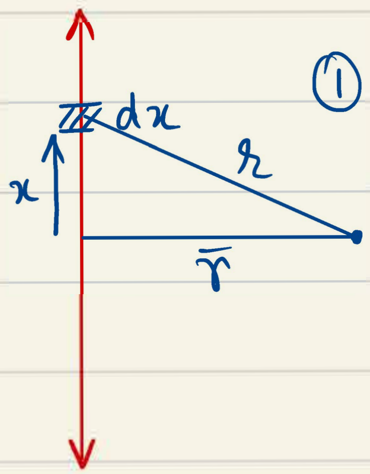


$$V(\bar{r}) = -\int_{-\infty}^r \frac{Q}{4\pi\epsilon_0} \frac{dr'}{r'^2} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r'}\bigg|_\infty^r = \frac{Q}{4\pi\epsilon_0 r} \quad (r \geq R)$$

For $r < R$: $V(\bar{r}) = \underbrace{-\int_P^R \bar{E} \cdot dl'}_{\text{outside}} \underbrace{- \int_R^r \bar{E} \cdot dl'}_{\text{Inside}}$

$$V(\bar{r}) = \frac{Q}{4\pi\epsilon_0 R} \quad (r < R)$$

Ex: Potential of an infinitely long charge distribution.



① $$V(\bar{r}) = \int_{x=-\infty}^{x=\infty} \frac{\lambda \, dx}{4\pi\epsilon_0 \, r} = \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dx}{(r^2+x^2)^{1/2}}$$

$$\int_0^\infty \times 2$$

→ We have a problem!

$$\ln\left(\frac{x}{r} + \sqrt{1 + x^2/r^2}\right)\Bigg|_{-\infty}^{\infty}$$

② $$V(\bar{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\bar{l} = \int \frac{-\lambda}{2\pi\epsilon_0 r'} \, dr' = -\frac{\lambda}{2\pi\epsilon_0} \ln r' \Bigg|_{\infty}^{r}$$

V for $r \to \infty$ is unknown ∵ the charge distribution itself extends to ∞.

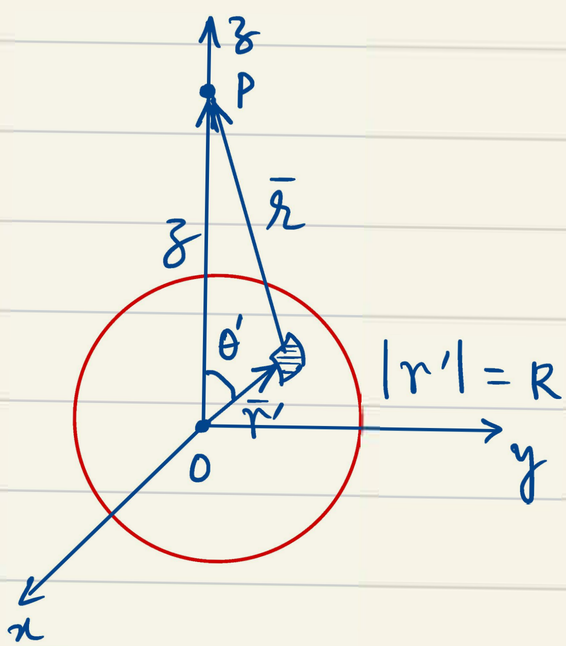
∴ We write $$V(\bar{r}) = -\frac{\lambda \ln r}{2\pi\epsilon_0} + K$$

$$V(\bar{r}) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r}$$

For localised and Finite Charge distributions :-

$\rho(\bar{r})$	$\sigma(\bar{r})$	$\lambda(\bar{r})$
$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar{r}')}{\mathscr{r}} d\tau'$	$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\bar{r}')\,da'}{\mathscr{r}}$	$= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\bar{r}')dl'}{\mathscr{r}}$

Ex: Electrostatic Potential of a spherical shell.



$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(r')\,da'}{\mathscr{r}}$$

$$= \frac{\sigma}{4\pi\epsilon_0}\int \frac{R^2 \sin\theta'\,d\theta'\,d\varphi'}{(R^2+z^2-2Rz\cos\theta')^{1/2}}$$

$$= \frac{2\pi R^2\sigma}{4\pi\epsilon_0}\int_0^{\pi} \frac{\sin\theta'\,d\theta'}{(R^2+z^2-2Rz\cos\theta')^{1/2}}$$

$$= \frac{R^2\sigma}{2\epsilon_0}\left[\frac{1}{Rz}(R^2+z^2-2Rz\cos\theta')^{1/2}\right]_0^{\pi}$$

$$V(\bar{r}=z) = \frac{R\sigma}{2\epsilon_0 z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

Note: z above is an arbitrary point on z axis

Case I: $z > R$ (Outside)

$$V(\bar{r}) = \frac{R\sigma}{2\epsilon_0 z}\left[R+z-(z-R)\right] = \frac{R\sigma}{2\epsilon_0 z} \times 2R = \frac{\sigma R^2}{\epsilon_0 z}$$

$$= \frac{\sigma R^2}{\epsilon_0 z} \times \frac{4\pi}{4\pi} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$ (r = distance from centre of sphere)

Case II: $z < R$ (Inside)

$$V(\bar{r}) = \frac{R\sigma}{2\epsilon_0 z}\left[R+z-(R-z)\right] = \frac{R\sigma}{2\epsilon_0 z} \times 2z = \frac{\sigma R}{\epsilon_0}$$

$$= \frac{\sigma R}{\epsilon_0} \times \frac{4\pi R}{4\pi R} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$

$\uparrow$

The potential inside the hollow charged sphere is a constant

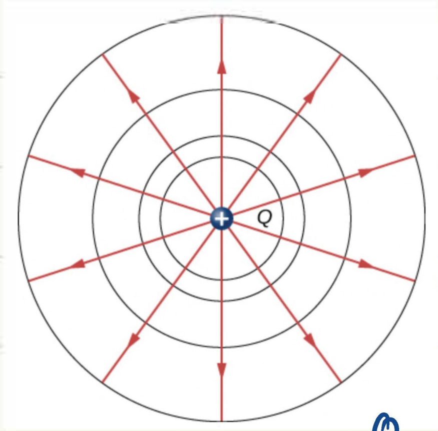
$\Rightarrow \bar{E} = 0 \quad \because -\bar{\nabla}V = 0$

Equipotentials

Equipotentials are imaginary surfaces or lines that have the same potential. i.e. $V(\bar{r}) = \text{Const.}$

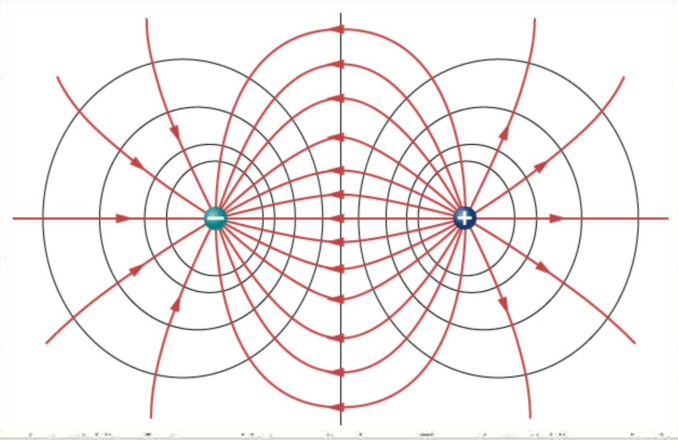
$\equiv$ isotherms / isobars etc.

Equipotentials of some charge distributions

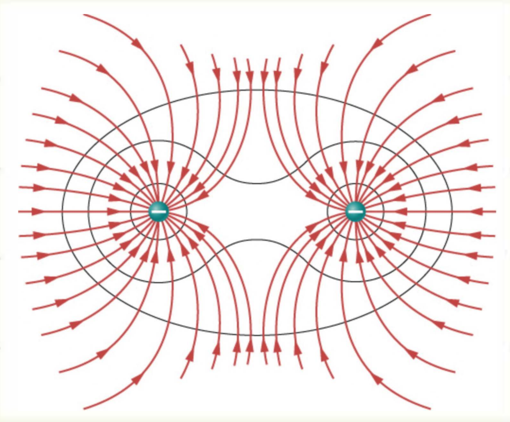


isolated point charge

EQN: $r = \dfrac{Q}{4\pi\varepsilon_0 V}$



+q/−q dipole



2 −q charges

Regions with closely placed equipotentials are regions of strong $\bar{E}$

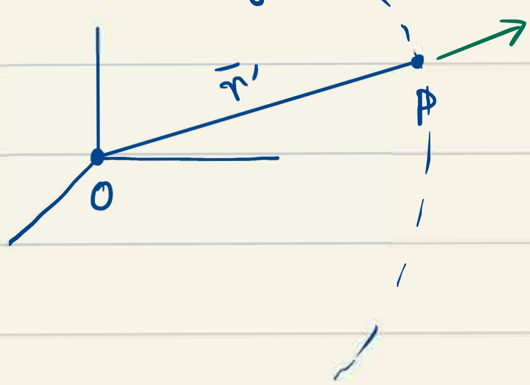
$\because \vec{E} = -\bar{\nabla} V$

Look back at all the Equations we got.

$\rho(\bar{r})$	$\bar{E}(\bar{r})$	$V(\bar{r})$
$\vec{E}(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar{r}')}{\mathscr{r}'^{2}}\, d\tau'$	$\rho = \epsilon_0 \bar{\nabla}\cdot\bar{E}$	$\rho = -\epsilon_0 \nabla^2 V$
$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar{r}')}{\mathscr{r}'}\, d\tau'$	$V = -\int_P^r \vec{E}\cdot d\mathbf{l}'$	$\bar{E} = -\bar{\nabla} V$

But as you have encountered the above alone doesnot guarantee a solution! The boundary Conditions are of crucial importance.

Ex: <u>All</u> space is filled with uniform charge density ρ.

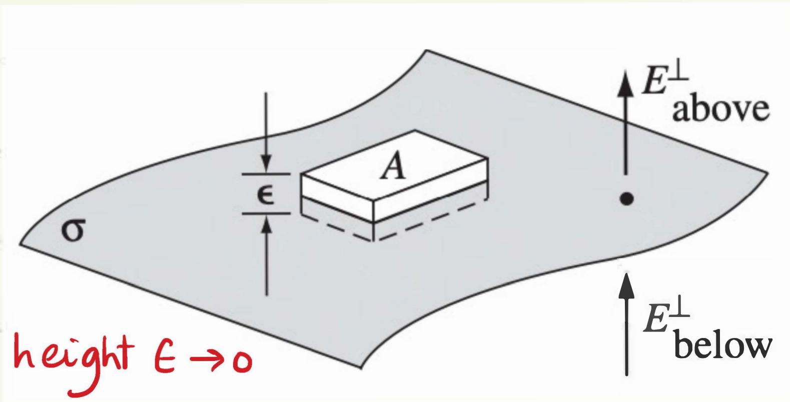


$$E \cdot 4\pi r^2 = \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0} \Rightarrow \bar{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$$

$$\nabla\cdot\bar{E} = \rho/\epsilon_0 \checkmark$$

$$\bar{\nabla}\cdot\bar{E} = \underline{\underline{0}}$$

Boundary Conditions



Consider an arbitary charged surface –

not necessarily a plane

σ is not uniform

$\bar{E}_{above}$ and $\bar{E}_{below}$ are the net $\bar{E}$ fields due to ALL sources

Now consider a Gaussian Surface as shown.

→ height of Gaussian Surface $\epsilon \to 0$

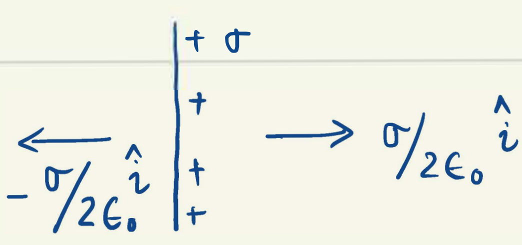
→ area A is small but finite: $E_\perp$ is uniform across A.

$$\oint \vec{E}\cdot d\bar{a} = \frac{\sigma A}{\epsilon_0} \Rightarrow \underset{(\text{above})}{E_\perp A} - \underset{(\text{below})}{E_\perp A} + \underbrace{E_{\parallel} A'}_{=\,0} = \frac{\sigma A}{\epsilon_0} \Rightarrow E_\perp = \frac{\sigma A}{\epsilon_0}$$

$\because A' \to 0$ but $E_{\parallel} \neq 0$

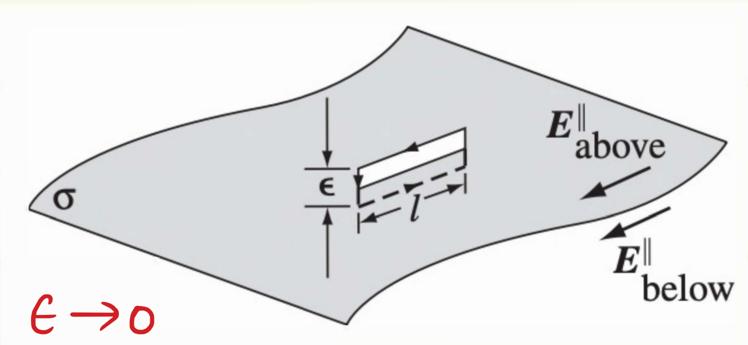
$$\Rightarrow E_\perp(\text{above}) - E_\perp(\text{below}) = \sigma/\epsilon_0$$

⇒ ⊥'r component of $\vec{E}$ field is always discontinuous across a boundary with surface charge density.



$$E_\perp^{(R)} - E_\perp^{(L)} = \frac{\sigma}{2\epsilon_0} - \left(-\frac{\sigma}{2\epsilon_0}\right) = \sigma/\epsilon_0$$

What about the $E_{\parallel}$ across the surface?



$\epsilon \to 0$

We know that:

$$\oint \vec{E} \cdot d\bar{l} = 0$$

l is small but finite

$$\Rightarrow \int_1 \bar{E} \cdot d\underset{\uparrow}{\bar{l}} + \int_2 \bar{E} \cdot d\underset{\uparrow}{\bar{l}} + \int_3 \bar{E} \cdot d\underset{\uparrow}{\bar{l}} + \int_4 \bar{E} \cdot d\underset{\uparrow}{\bar{l}} = 0$$

$l \qquad \epsilon \qquad l \qquad \epsilon$

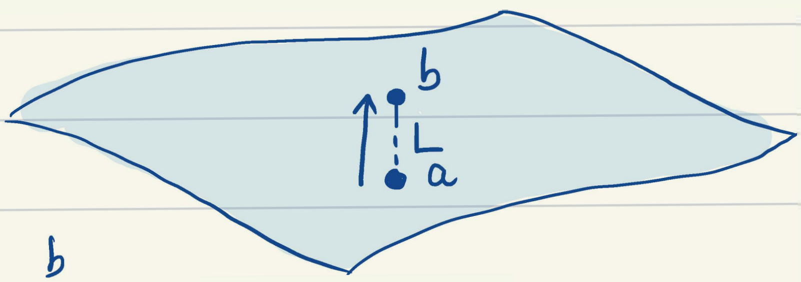
$$\Rightarrow \quad E_{\parallel}(\text{above})\, l + 0 - E_{\parallel}(\text{below})\, l + 0 = 0$$

$$\Rightarrow \quad E_{\parallel}(\text{above}) = E_{\parallel}(\text{below})$$

$\Rightarrow$ $\parallel$ Component of $\vec{E}$ field is always continuous across a boundary with surface charge density.

$$\Rightarrow \quad \vec{E}(\text{above}) - \vec{E}(\text{below}) = \frac{\sigma}{\epsilon_0}\hat{n}$$

If you integrate $-\int_a^b \bar{E}.\bar{dl}$ along a line L ⊥'r to the surface.



$$\Rightarrow \lim_{L\to 0} -\int_a^b \vec{E}.\vec{dl} = 0 = V(b) - V(a)$$

$$\Rightarrow \quad V(\text{above}) = V(\text{below})$$

$V(\bar{r})$ is continuous across the interface with surface charge

$$\because \quad E_\perp(\text{above}) - E_\perp(\text{below}) = \sigma/\epsilon_0$$

$$\Rightarrow \frac{\partial V}{\partial n}(\text{above}) - \frac{\partial V}{\partial n}(\text{below}) = -\sigma/\epsilon_0$$

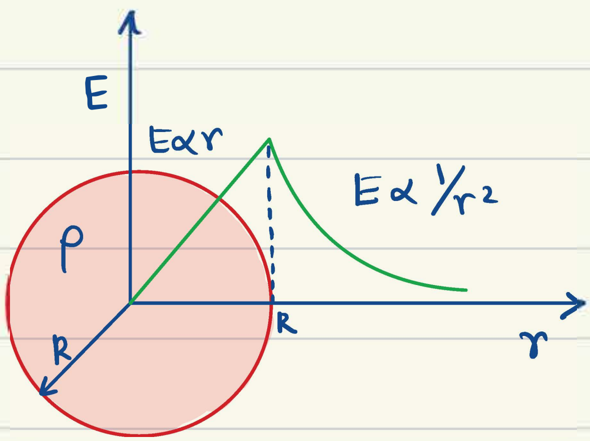
$\bar{\nabla} V . \hat{n}$

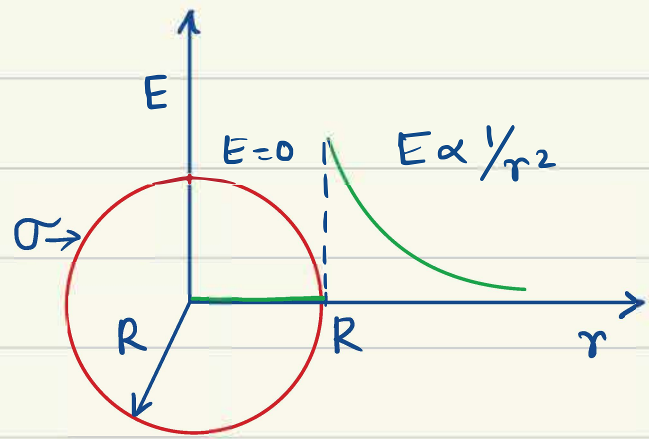
Note: Above are "local" conditions valid on any interface/surface with surface charge density.

A comparison: a sphere with charge density ρ

$$Q = \frac{4}{3}\pi R^3 \rho$$

a spherical shell with surface charge density σ $\quad Q = 4\pi R^2 \sigma$





No discontinuity in $E_\perp$ at sphere surface.

$$E_\perp(r=R+\delta) - E_\perp(r=R-\delta) = 0$$

$$\frac{\rho R}{3\epsilon_0} - \frac{\cancel{\frac{4}{3}}\cancel{\pi} R^{\cancel{3}}\rho}{\cancel{4}\cancel{\pi}\epsilon_0 \cancel{R^2}} = 0$$

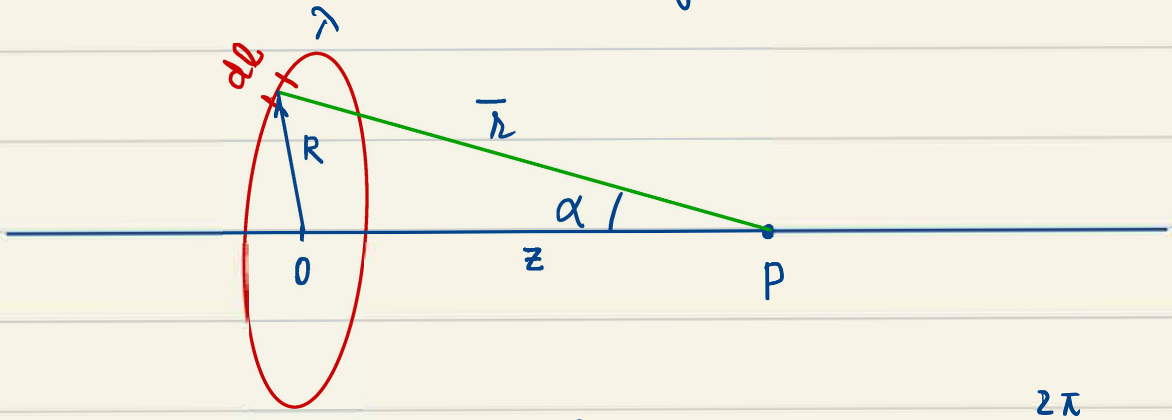
Discontinuity in $E_\perp$

$$E_\perp(r=R+\delta) - E_\perp(r=R-\delta)$$

$$\frac{Q}{4\pi\epsilon_0 R^2} - 0 = \frac{\cancel{4}\cancel{\pi}\cancel{R^2}\sigma}{\cancel{4}\cancel{\pi}\epsilon_0 \cancel{R^2}}$$

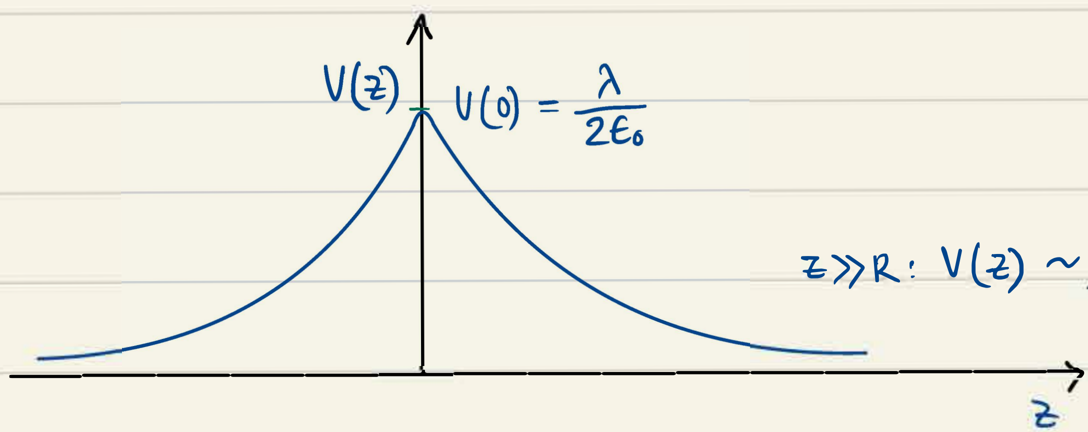
$$\Rightarrow E_\perp(\text{above}) - E_\perp(\text{below}) = \sigma/\epsilon_0$$

Ex: Potential of a ring of charge density λ C/m.



$$V(z) = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{(z^2+R^2)^{1/2}} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda R\, d\theta}{(z^2+R^2)^{1/2}} = \frac{\lambda R}{2\epsilon_0 (z^2+R^2)^{1/2}}$$

Note: $V(z)$ is an even function of $z \Rightarrow V(-z) = V(z)$



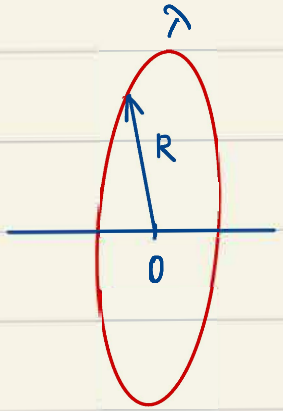
$z \gg R$: $V(z) \sim \dfrac{\lambda R}{2\epsilon_0 z}$ [ring behaves like a pt charge]

$z \ll R$ $\quad V(z) = \dfrac{\lambda R}{2\epsilon_0 R}\dfrac{1}{(1+z^2/R^2)^{1/2}} = \dfrac{\lambda}{2\epsilon_0}\left\{1 - \dfrac{z^2}{2R^2} + \cdots\right\}$

$$V(z) \simeq \frac{\lambda}{2\epsilon_0} - \frac{\lambda z^2}{2\epsilon_0 R^2}$$

$$\frac{\partial V}{\partial z} = -\frac{\lambda z}{\epsilon_0 R^2} \quad \Rightarrow \quad V(z) \text{ has a maxima at } z=0$$

Ex: What if a -ve point charge, constrained to move along a line $\perp$'r to the plane of the circle and is left to interact with the +vely charged circle?



What is the potential "energy landscape" encountered by the -ve point charge?

(ρ, θ, z) $\qquad$ $V(z)$ is known

$V(\rho, \theta, z)$?

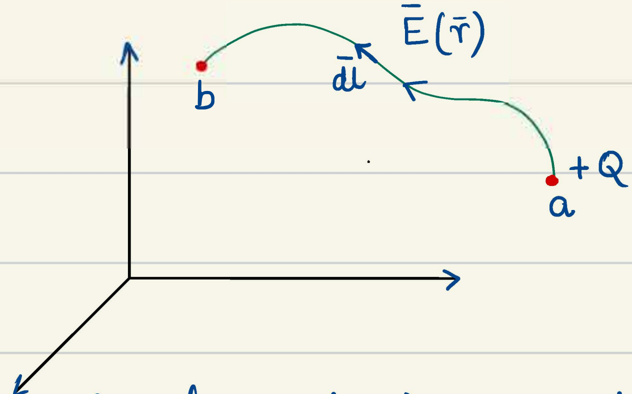
Work and Energy

Note: The Electric Field is a conservative Force field

$\Rightarrow$ WD in moving a charged particle is independent of path.

$\Rightarrow$ KE + PE = Constant

$\vec{E} = -\bar{\nabla} V, \quad \bar{\nabla} \times \bar{E} = 0$



$$\bar{F}(\bar{r}) = Q\bar{E}(\bar{r})$$

Work done in moving the charge Q from point $a \to b$

$$W = \int_a^b F(\bar{r}) \cdot d\bar{l} = -\int_a^b Q\,\bar{E}(\bar{r}) \cdot d\bar{l} = -Q \int_a^b \vec{E} \cdot d\bar{l}$$

Force exerted by you to move the charge quasistatically!

$$W_{ab} = Q\left[V(b) - V(a)\right]$$

$\Rightarrow$ The potential difference between two points is equal to the work done in moving a unit +ve charge between the points.

Work Done in Assembling a Charge distribution.

Say we want to assemble a charge distribution piece by piece, like assembling with lego blocks.

Start with empty space: $E(\bar{r}) = 0 \quad V(\bar{r}) = 0$

Bring in charge q_1 to point $\bar{r}_1$: Work done $= W_1 = 0$

$$\bar{r}_1 \longleftarrow \infty$$

Bring in charge q_2 to point $\bar{r}_2$:

$$W_2 = \int_{\infty}^{r_2} q_2 \frac{q_1}{4\pi\epsilon_0 r'} dr'$$

$$= \frac{q_2 q_1}{4\pi\epsilon_0 r_{12}} \qquad r_{12} = |\bar{r}_1 - \bar{r}_2|$$

Bring in charge q_3 to point $\bar{r}_3$:

$$W_3 = \frac{q_3 q_1}{4\pi\epsilon_0 r_{13}} + \frac{q_3 q_2}{4\pi\epsilon_0 r_{23}}$$

Total $W = W_1 + W_2 + W_3 + \ldots\ldots$

$$W = \frac{1}{2} \times \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{q_i q_j}{r_{ij}} \quad , \quad r_{ij} = |\bar{r}_i - \bar{r}_j|$$

Note: all $i = j$ are to be avoided which are "self interaction terms".

$\because$ both sums i & j $= 1 \rightarrow n$

The factor $\frac{1}{2}$ ensures that each pair is counted once.

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j \neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\bar{r}_i)$$

potential at the final position of q_i due to all other charges.

The order in which the charges were assembled does not matter

If $W = \frac{1}{2}\sum q_i V_i(\bar{r}_i)$ is the WD for assembling

Then the WD for dis-assembling $= -W$

What about continuous charge distributions?

$$W = \frac{1}{2}\sum_i q_i V_i \longrightarrow \frac{1}{2}\int \rho V d\tau' \text{ or } \frac{1}{2}\int \sigma V da' \text{ or } \frac{1}{2}\int \lambda V dl'$$

Volume of charge distribution

$$W = \frac{\epsilon_0}{2}\int_{vol} (\bar{\nabla}.\bar{E}) V d\tau$$

$$= \frac{\epsilon_0}{2}\int \left(\nabla.(\bar{E}V) - \bar{E}.\bar{\nabla}V\right) d\tau$$

$$= \frac{\epsilon_0}{2}\int \nabla.(\bar{E}V) d\tau + \frac{\epsilon_0}{2}\int \bar{E}.\bar{E} \, d\tau$$

$$= \frac{\epsilon_0}{2}\int_S \bar{E}V.d\bar{a} + \frac{1}{2}\epsilon_0 \int_{vol} E^2 d\tau$$

The integration is over the volume of the charge distribution — But can be extended over all space $\because \rho(\bar{r})$ will be zero elsewhere

For any finite charge distribution $\int_S \bar{E}V.d\bar{a} \rightarrow 0$ as $r \rightarrow \infty$

$$\boxed{W = \frac{1}{2}\epsilon_0 \int E^2 d\tau \quad \text{Vol} = \text{all space}}$$

It seems to make sense but does it?
Remember WD in bringing in the 1st charge

$W_1 = 0$

What is the Energy stored in the field of a single point charge?

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2} \int \frac{Q^2}{r^{\cancel{4}\,2}} \cancel{r^2} \sin\theta \, d\theta \, d\phi \, dr = \frac{Q^2}{8\pi\epsilon_0} \int_0^\infty \frac{1}{r^2} dr$$
$$= \infty$$

This gives the self energy of a point charge.
Which is infinite !!

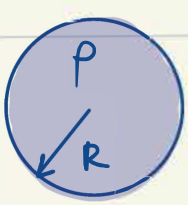
→ Limitation of the concept of a point charge.

→ Cannot resolve within the framework of Classical EM

Remember: No problem with $W = \frac{\epsilon_0}{2} \sum_i q_i V(\bar{r}_i)$

self term $i=j$ excluded in the statement

If you replace the point charge with a sphere of charge density of finite radius R and calculate



$$\frac{\epsilon_0}{2} \int E^2 d\tau = \frac{\epsilon_0}{2} \left[\int E^2_{inside} \, d\tau + \int E^2_{outside} \, d\tau \right]$$
$$= \text{FINITE}$$ (HOME WORK)

Note :-

1. Energy calculated corresponds to the whole system charges + field.

2. Superposition principle does not hold for calculating the total Energy.

3. Starting point for calculating the stability of molecules, compounds etc.

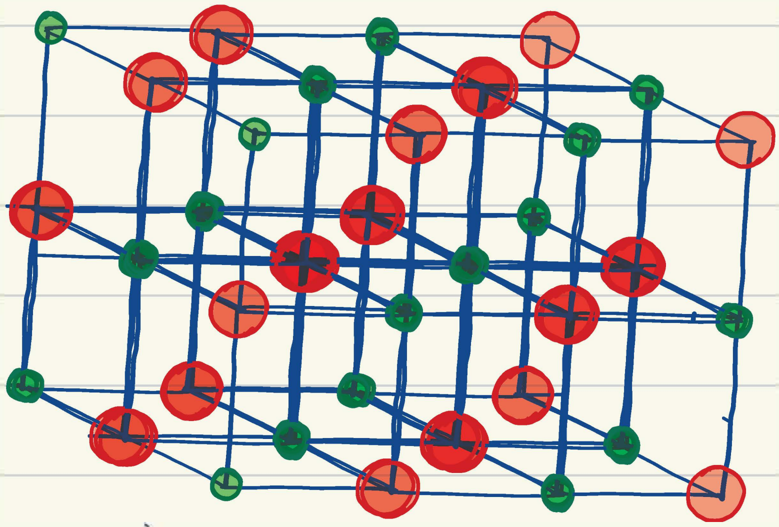
4. $$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau$$

Energy density W/unit volume $= \epsilon_0 E^2/2$

The Madelung Constant

Determine the Electrostatic Potential Energy of an ion in a crystal. (see wikipedia page)

Ex: NaCl

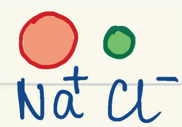


Cubic Lattice of spacing

$a \sim 2.81\,\text{Å}$

the PE = Energy required to dissociate the Xtal into ions.

= 7.92 eV/molecule



Can we show this?

Calculate the P.E. per ion. ×2 = P.E. per molecule.

Identify a Na^+ ion in a chain.

.... Na^+ Cl^- Na^+ Cl^- Na^+ Cl^- ....

$$V_1 = \frac{e^2}{4\pi\epsilon_0}\left[\cdots - \frac{1}{3a} + \frac{1}{2a} - \frac{1}{a} + -\frac{1}{a} + \frac{1}{2a} - \frac{1}{3a} + \frac{1}{4a} \cdots\right]$$

$$= \frac{e^2}{4\pi\epsilon_0 a}\left[-\frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \cdots\right]$$

$$V_1 = -\frac{2e^2}{4\pi\epsilon_0 a}\ln 2 = -1.386\,e^2/4\pi\epsilon_0 a = -1.386 \times 5.12\,\text{eV}$$

Now calculate the P.E. of that ion due to nearest neighbour line of $Na^+ Cl^- Na^+ Cl^-$ — there are 4 of them

Then calculate for next nearest neighbour line. an so on.

$$\sum V_i \simeq \frac{-1.747}{4\pi\epsilon_0 a} = -8.94\,eV$$

For a xtal in general Potential Energy of i^{th} ion

$$V_i = \frac{e}{4\pi\epsilon_0 r_0} \sum_j \frac{z_j r_0}{r_{ij}}$$

Charge

nearest neighbour

$$= \frac{e}{4\pi\epsilon_0 r_0} M_i$$

Madelung Constant.

NaCl: 2 cubic lattices of Na^+ and Cl^- intersect.

$$M_{Na^+} = M_{Cl^-} = -1.74756\ldots\ldots$$

Basic Assumption: Ions can be approximated as point charges with spherical symmetry

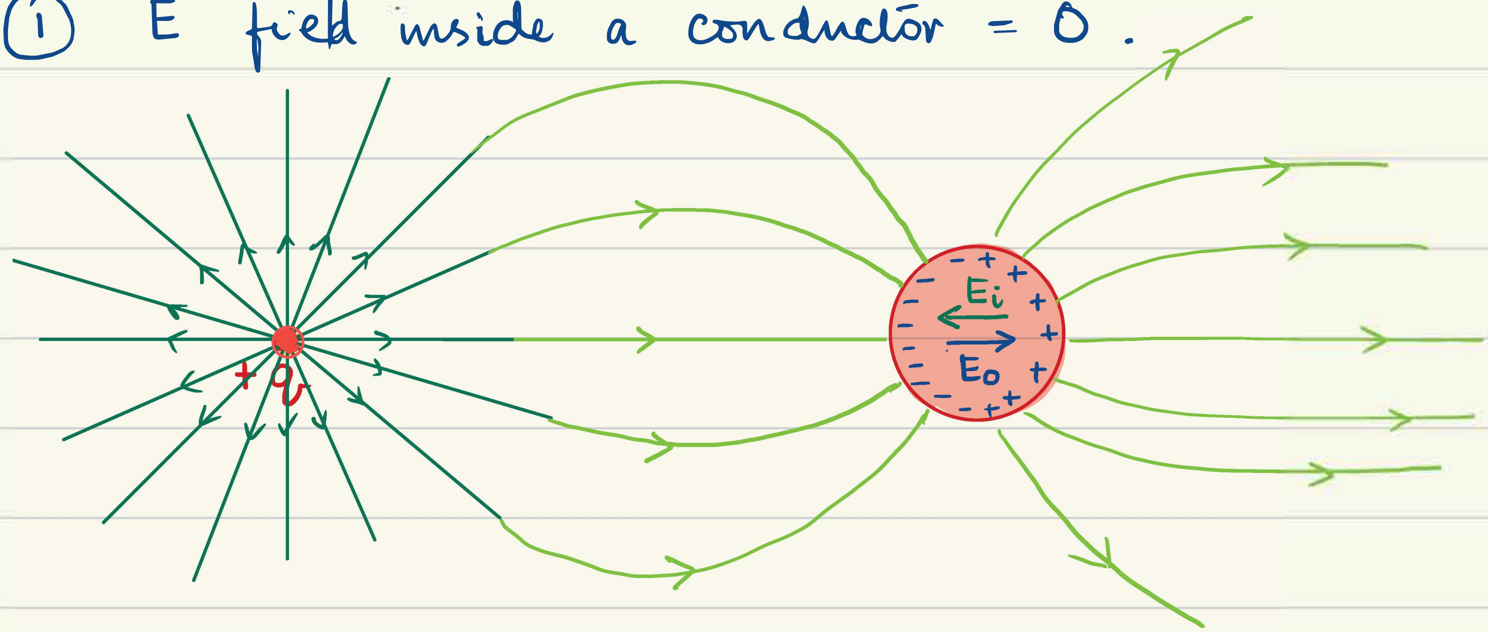
Conductors

Perfect conductors:
Idealized material with ∞ supply of free electrons.

Real Conductors
Gold / Silver / Al / Cu etc.
e^- density $n \sim 10^{28}/m^3$

Properties:

① $\vec{E}$ field inside a conductor $= 0$.



To what extent will the free charges polarize?

Original Field due to $+q$: $\bar{E}_0$
Induced Field to polarization : $\bar{E}_i$

Inside the sphere
$\bar{E}_{NET} = \bar{E}_0 + \bar{E}_i = 0$

What is the $\vec{E}$ field inside the conductor?

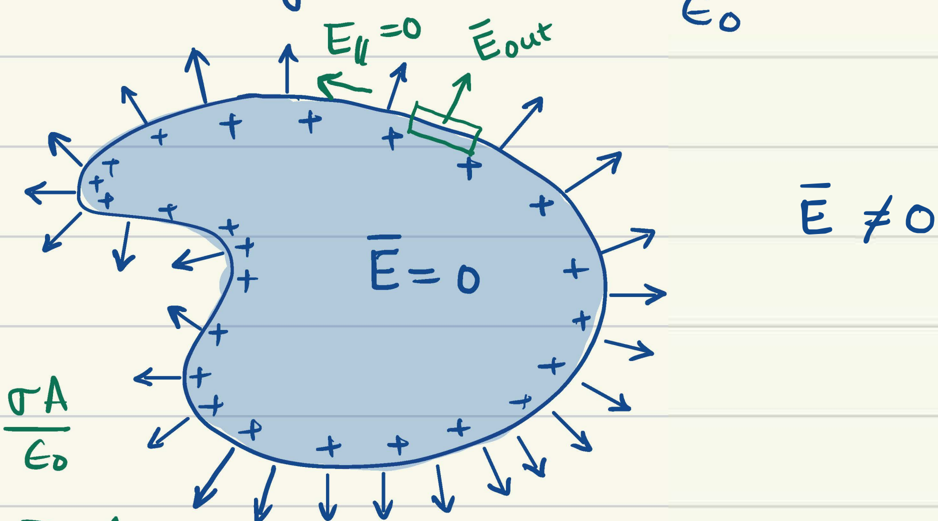
② $\rho = 0$ inside

$\bar{\nabla} \cdot \bar{E} = \rho/\epsilon_0$ if $\bar{E}_{inside} = 0 \Rightarrow \rho = 0$

③ All net charges reside at the surface, bulk $= 0$

④ $\because \bar{E}_{inside} = 0 \Rightarrow$ a conductor is an Equipotential.

⑤ $\bar{E}$ at surface of a charged conductor is always $\perp$'r to surface locally and $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$



$\oint E \cdot da = \dfrac{\sigma A}{\epsilon_0}$

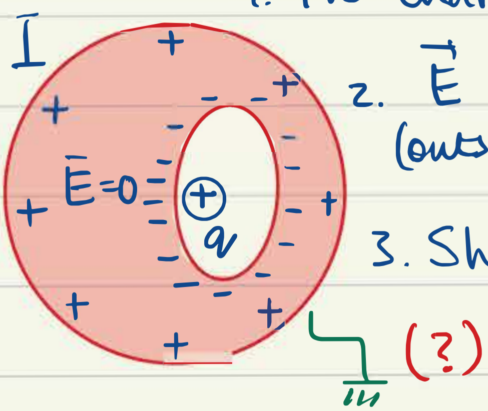
$E_{\perp}$ outside A $+ \cancel{E_{\perp} \text{inside} A}^{\,0} = \dfrac{\sigma A}{\epsilon_0}$

$\Rightarrow \bar{E}_{\perp}$ outside $= \dfrac{\sigma}{\epsilon_0}\hat{n}$

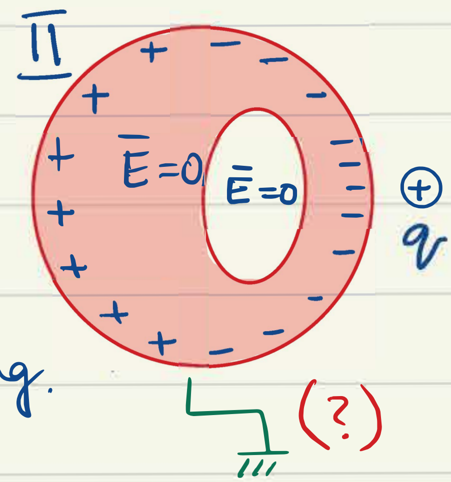
$E_{\parallel}$ outside $= 0$

Note: The above are intricately related to $1/r^2$ nature of Coulomb's Law and 3D world.

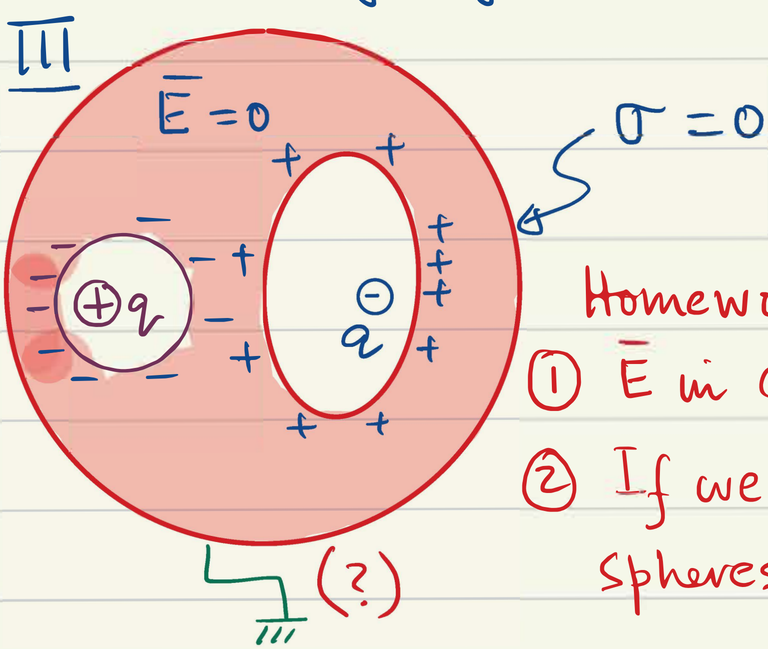
Cavity in a conductor



1. +ve charge distributed uniformly across outer surface
2. $\vec{E}_{(outside)} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ r: distance from ⊙ of sphere
3. Shape/location of cavity + charge does not matter



1. The Field in the cavity $\bar{E}=0$!!
2. The charges on the conductor distribute in a way to make $\bar{E}_{inside} = 0$.
3. Faraday Cage – Electrostatic Shielding.

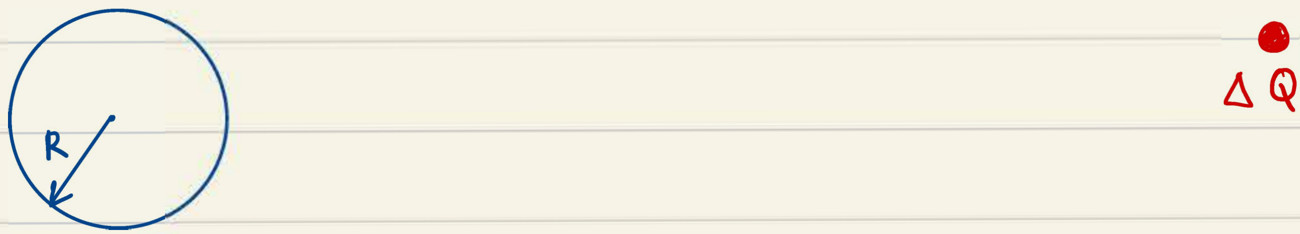


Homework:

① $\bar{E}$ in conductor $= 0 \Rightarrow V = \text{Constant}$ (?)

② If we ground these conducting spheres. What happens?

Ex: What is the work done in charging an isolated spherical conductor of Radius R to charge Q_0?



→ Assume that we bring in an infinitsimal amont ΔQ at a time from ∞, and put on the conducting sphere. Lets say that many such trips have happened and the presently the charge on the sphere is Q $(Q < Q_0)$.

∴ the WD in bringing further ΔQ charge from ∞ to the surface of the sphere is.

$$\Delta W = \int_{\infty}^{R} \bar{F}.d\bar{r}' = -\int_{\infty}^{R} \Delta Q\, \bar{E}(\bar{r}').dr' = \int_{\infty}^{R} \frac{-Q\,\Delta Q}{4\pi\epsilon_0 r'^2} dr' = \frac{Q\,\Delta Q}{4\pi\epsilon_0 R}$$

We can further integrate both sides above to get the total WD.

$$\Rightarrow \quad W = \frac{Q_0^2}{8\pi\epsilon_0 R}$$ = Energy stored in the isolated conducting sphere with charge Q_0

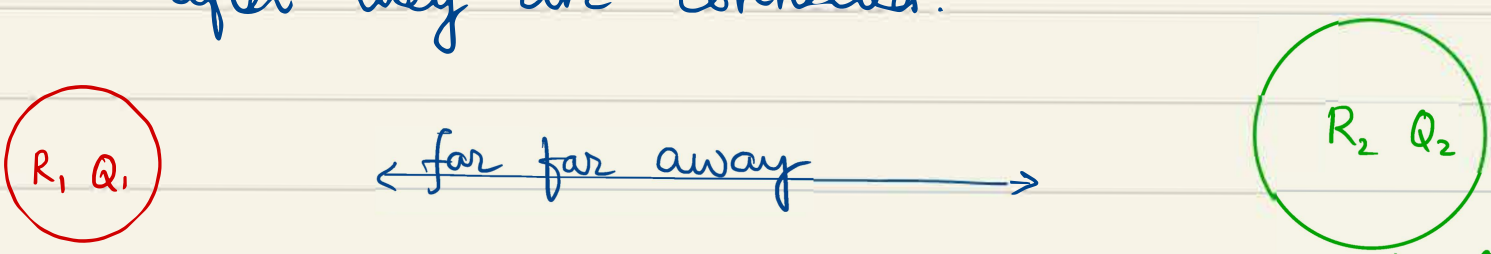
$$V \text{ of sphere} = \frac{1}{4\pi\epsilon_0}\frac{Q_0}{R}$$

The isolated sphere is also a capacitor of Capacitance C where $E = \frac{1}{2} CV^2 = \frac{1}{2} QV = \frac{1}{2} Q^2/C$

$$\Rightarrow \quad E = \frac{1}{2} . Q . \frac{Q}{4\pi\epsilon_0 R} = \frac{1}{2} QV \quad \text{with } V = \frac{Q}{4\pi\epsilon_0 R}$$

and $\boxed{C = 4\pi\epsilon_0 R}$ ← Capacitance of an isolated charged sphere.

Ex: Consider 2 isolated conducting spheres of radii R_1 and R_2 and carrying charges Q_1 and Q_2. They are now connected by a conducting wire. Calculate the charges on the spheres after they are connected.



$V_1 = \frac{1}{4\pi\epsilon_0} \frac{Q_1}{R_1}$

$V_2 = \frac{1}{4\pi\epsilon_0} \frac{Q_2}{R_2}$

Total charge $Q = Q_1 + Q_2$

At Equillibrium the potential at both sphere are Equal

$$V = \frac{1}{4\pi\epsilon_0} \frac{Q_1'}{R_1} = \frac{1}{4\pi\epsilon_0} \frac{Q_2'}{R_2} \Rightarrow \quad Q_1' R_2 = Q_2' R_1$$

and $\sigma_1' R_1 = \sigma_2' R_2$

$$\Rightarrow \quad Q = Q_1 + Q_2 = Q_1' + Q_2'$$

$$\Rightarrow Q_2' = Q_1 + Q_2 - Q_1' = Q_1' \frac{R_2}{R_1}$$

$$\Rightarrow \left(1 + \frac{R_2}{R_1}\right) Q_1' = Q_1 + Q_2 \Rightarrow Q_1' = \frac{R_1 (Q_1 + Q_2)}{(R_1 + R_2)}$$

$$\Rightarrow Q_2' = \frac{R_2 (Q_1 + Q_2)}{(R_1 + R_2)}$$

if $R_1 < R_2 \Rightarrow Q_1' < Q_2'$

What about the $\vec{E}$ field near the 2 spheres?

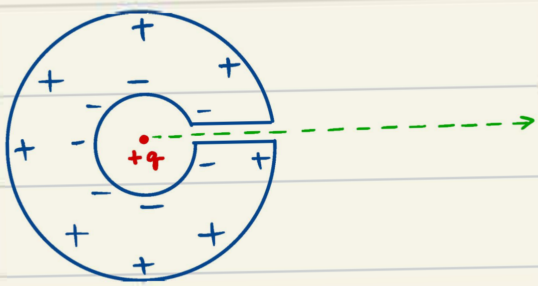
At the surface of sphere R_1: $\bar{E}_1' = \sigma_1'/\epsilon_0 \, \hat{r}$

surface of sphere R_2: $\bar{E}_2' = \frac{\sigma_2'}{\epsilon_0} \hat{r}$

$\because \quad \sigma_1' R_1 = \sigma_2' R_2$ and $R_1 < R_2 \Rightarrow \sigma_1' > \sigma_2'$

$\Rightarrow \quad E_1' > E_2' \leftarrow$ smaller sphere has higher surface charge density and local field.

Ex: Calculate the WD in removing a charge q from a cavity inside a conductor.



Chapter 3 Special Techniques in Electrostatics

Advanced Electrostatics – Special Techniques

Ruled by 3 functions: $\rho(\bar{r})$ $V(\bar{r})$ $\bar{E}(\bar{r})$

Governing Equation: $\bar{\nabla}\cdot\bar{E}(\bar{r}) = \dfrac{\rho(\bar{r})}{\epsilon_0}$ — ①

or $\nabla^2 V(\bar{r}) = -\dfrac{\rho(\bar{r})}{\epsilon_0}$ — ②

∇^2 ↑ Laplacian

$$\nabla^2 V(\bar{r}) = -\frac{\rho(\bar{r})}{\epsilon_0} \quad \leftarrow \text{Poisson Equation.}$$

Remember: $\nabla^2\left[V(\bar{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{\text{VOL}} \frac{\rho(\bar{r}')\,d\tau'}{r}\right] \equiv \dfrac{\rho(\bar{r})}{\epsilon_0}$ (PROVE)

$$\nabla^2\left(\frac{1}{|\bar{r}-\bar{r}_0|}\right) = -4\pi\,\delta(\bar{r}-\bar{r}_0)$$

Very often we want to calculate $V(\bar{r})$ and $\rightarrow \bar{E}(\bar{r})$ in regions that have no charge i.e. $\rho(\bar{r}) = 0$

$\Rightarrow$ Eqn ② : $\nabla^2 V(\bar{r}) = 0$ ← Laplaces Equation

↑ Satisfied in all space without charge.

In Cartesian Co-ords: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

An 1D world:

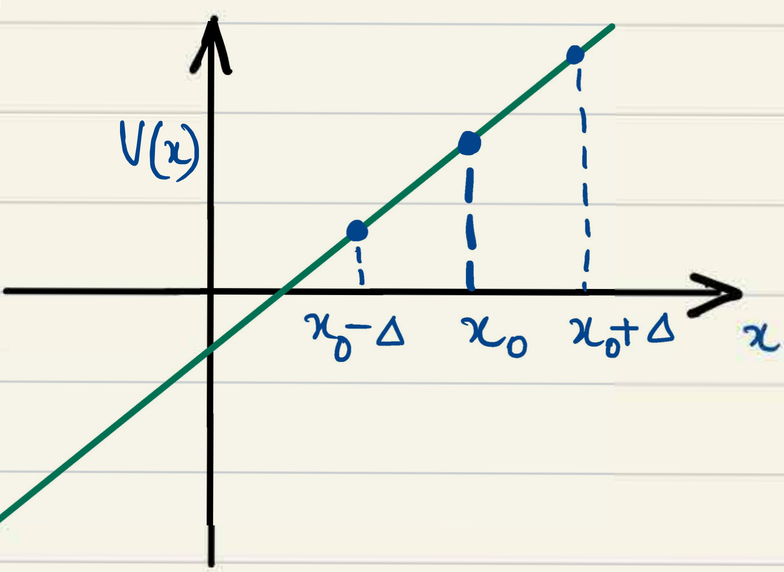
$$\nabla^2 V(x) = \frac{d^2 V}{dx^2} = 0$$

$\Rightarrow \frac{dV}{dx} = k \Rightarrow V(x) = kx + m$ (Eqn. of a straight line)

But we donot know what is "k" and "m"!

Need $\boxed{\text{Boundary Conditions}}$ to determine $V(x)$ fully.

Property 1:



$\because$ $V(x)$ is a straight line

Slope is a constant

$$\Rightarrow V(x_0) = \frac{V(x_0+\Delta) + V(x_0-\Delta)}{2}$$

for any value of Δ.

Property 2:

In the space where $\nabla^2 V = 0$, V has no local maxima or minima.

In 2D: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

In 3D: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

- No local maxima or minima
  - $\Rightarrow$ A purely Electrostatic configuration can never hold a charge particle in stable equilibrium. ← Earnshaw's Theorem

$\nabla^2 V = 0 \Rightarrow$ ALL minima excluded.

The min/max values of V are at the boundaries.

- $V(\bar{r})$ is always equal to an average of values around it.

2D: $V(\bar{r}) = \frac{1}{2\pi R} \int_{\text{Circle}} V\, dl$ 3D: $V(\bar{r}) = \frac{1}{4\pi R^2} \int_{\text{Sphere}} V\, da$

For any value of R!

General Solution of V satisfying $\nabla^2 V = 0$

(Set of functions called Harmonic Fns)

$+$

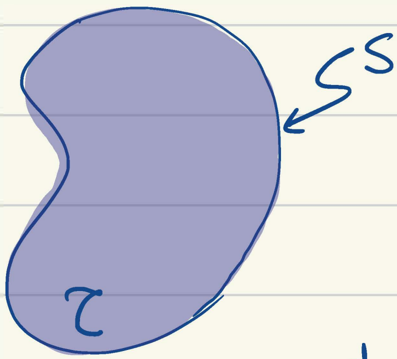
Adequate number of Boundary Conditions

$\Downarrow$

Particular Solution of V.

Uniqueness Theorem 1: (as discussed in Griffiths)

The solution to $\nabla^2 V = 0$ in a volume τ is uniquely determined if V is specified on the bounding surface S.



Specify V at all points on S $\Rightarrow$ V at all points in τ is uniquely determined from $\nabla^2 V = 0$

Proof: Say the solution to $\nabla^2 V = 0$ is not unique

$\Rightarrow \nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$ 2 solutions for the same boundary condition.

let $V_3 = V_2 - V_1 \Rightarrow \nabla^2 V_3 = \nabla^2 V_2 - \nabla^2 V_1 = 0$

$\therefore V_3$ also satisfies Laplaces Eqn.

at the boundary we know $V_2 = V_1 \Rightarrow V_3 = 0$

$\because V_3$ does not have any local max. or min.

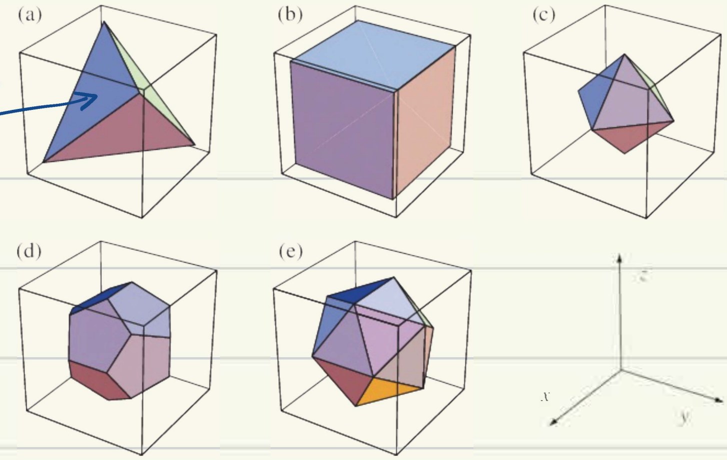
and $V_3 = 0$ at boundary $\Rightarrow V_3 = 0$ everywhere

$\Rightarrow V_1 = V_2$ at all points.

Corollary: The theorem is true even in the presence of charges i.e. Poisson's Eqn

Restate Th: The potential in any volume is uniquely determined if the charge density is known throughout the region and the potential at all points on the bounding surfaces are specified.

a_n



Fun Problem

Say I have an n-sided regular polyhedra $\{n = 4, 6, 8, 12, 20 \ldots\}$

The potential at each face is V_i $(i = 1, 2, 3, \ldots n)$

Prove that the potential at the centre is $\overline{V_i}$.

Assume one side has potential V_i and others zero.
Let The resulting potential at the Centre $= V_i'$
This can go on for one side at a time.

Using superposition principle $V_{centre} = \sum_{i=1}^{n} V_i'$

If all V_i are equal then all V_i' are also equal

$$\Rightarrow \quad V_{centre} = n V_i'$$

But if all V_i are same then potential at all points inside the polyhedron are also same. $\Rightarrow V_{centre} = V_i$

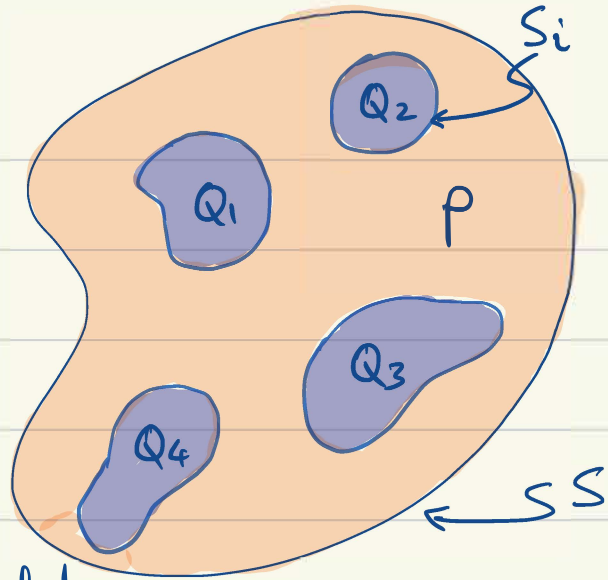
$\Rightarrow$ $V_i' = V_i/n$ is the contribution from one side.

$\therefore$ if V_i are different $\Rightarrow$ $V_{centre} = \sum_i V_i/n$

Say $n \to \infty$ and $a_n \to 0$ such that $n a_n = \text{const}\ (4\pi R^2)$

$$V_{centre} = \sum \frac{V_i a_i}{n a_i} \longrightarrow \frac{1}{4\pi R^2} \oint_S V \, da$$

Uniqueness Theorem 2:



In a volume τ containing conductors with known charges Q_i and charge density ρ, the Electric field is uniquely determined.

Proof: Say the solution is not unique

2 Fields $\bar{E}_1$ and $\bar{E}_2$ satisfy the boundary conditions

$\Rightarrow$	$\bar{\nabla}.\bar{E}_1 = \rho/\epsilon_0$	$\bar{\nabla}.\bar{E}_2 = \rho/\epsilon_0$	at all points not on conductors
	$\oint_{S_i} \bar{E}_1.d\bar{a} = Q_i/\epsilon_0$	$\oint_{S_i} \bar{E}_2.da = Q_i/\epsilon_0$	at the surface of all conductors
	$\oint_{S} \bar{E}_1.d\bar{a} = Q_T/\epsilon_0$	$\oint_{S} \bar{E}_2.da = Q_T/\epsilon_0$	at the bounding surface

again look at $\bar{E}_3 = \bar{E}_1 - \bar{E}_2$

Now $\bar{\nabla}\cdot\bar{E}_3 = 0$ in between conductors

and $\oint_{S_i} \bar{E}_3 \cdot d\bar{a} = 0$ at all conductor surfaces.

Let V_3 be the scalar potential corresponding to $\bar{E}_3$

ie. $\bar{E}_3 = -\bar{\nabla} V_3$. and $V_3 = V_1 - V_2$

V_3 is constant over the conductors.

Now consider the integral $\int_{VOL} \bar{\nabla}\cdot(\bar{E}_3 V_3)\, d\tau = \oint_{S+S_i} \bar{E}_3 V_3\, da$

$$\Rightarrow \int_V \cancel{(\bar{\nabla}\cdot\bar{E}_3)}^{\,0} V_3\, d\tau + \int_V \bar{E}_3\cdot(\bar{\nabla} V_3)\, d\tau = \oint_{S+S_i} \bar{E}_3 V_3\, da$$

$$\Rightarrow \quad -\int \bar{E}_3^2\, d\tau = \oint_{S+S_i} \bar{E}_3 V_3\, da$$

$$\left.\begin{array}{l}\text{at all } S_i: \quad V_3 = V_1 - V_2 = 0 \\ \text{and } S \to \infty \quad \text{where } V_3 = 0\end{array}\right\} \Rightarrow \int_V \bar{E}_3^2\, d\tau = 0$$

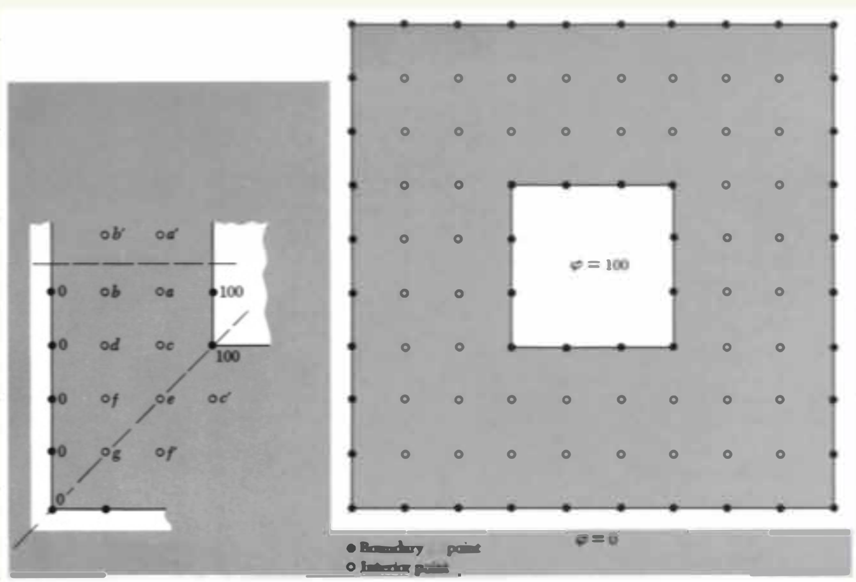
$\because$ E_3^2 is positive always

$\Rightarrow \bar{E}_3 = 0$

$\Rightarrow \bar{E}_2 = \bar{E}_1$!!

Crux of the Matter:

If you arrive at a solution for the potential or the field that satisfy the given boundary condition and charge density – then that is the correct solution – You can even guess the solution and you need not solve the problem at all.



Purcell (EM) Ch 3

Initial values

$a = c = e = 50$

$b = d = f = g = 25$

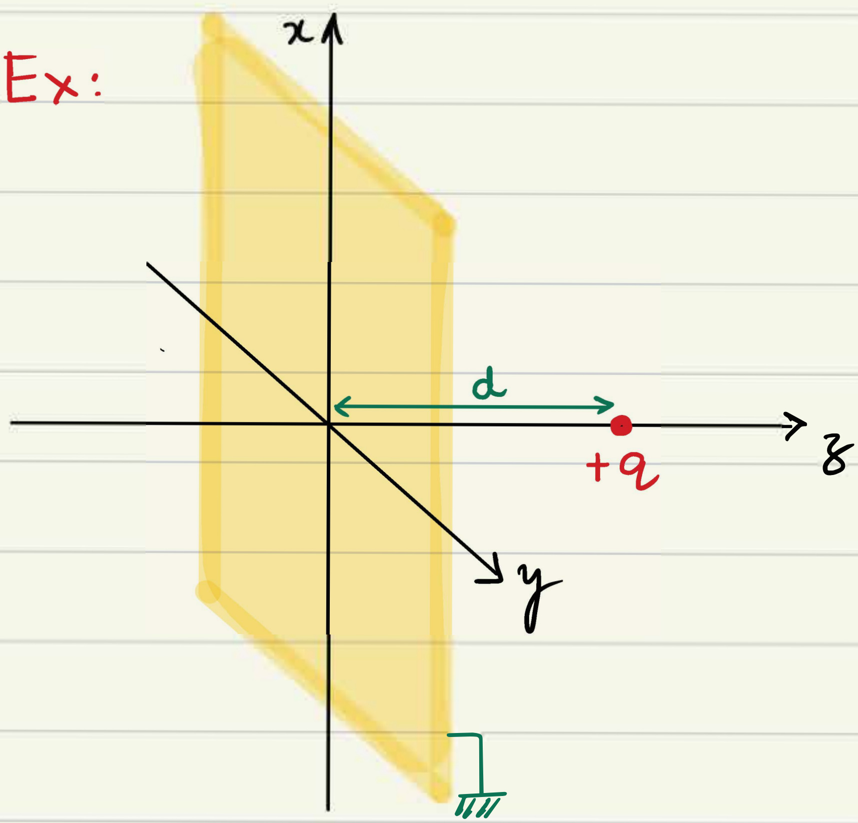
Update values by averaging with nearest neighbours

$a = (a' + 100 + c + b)/4$

$b = (b' + a + d + 0)/4$

Emboldened by the "Uniqueness Theorems" can we trick our way through calculating the potential for some charge distributions?

Ex:



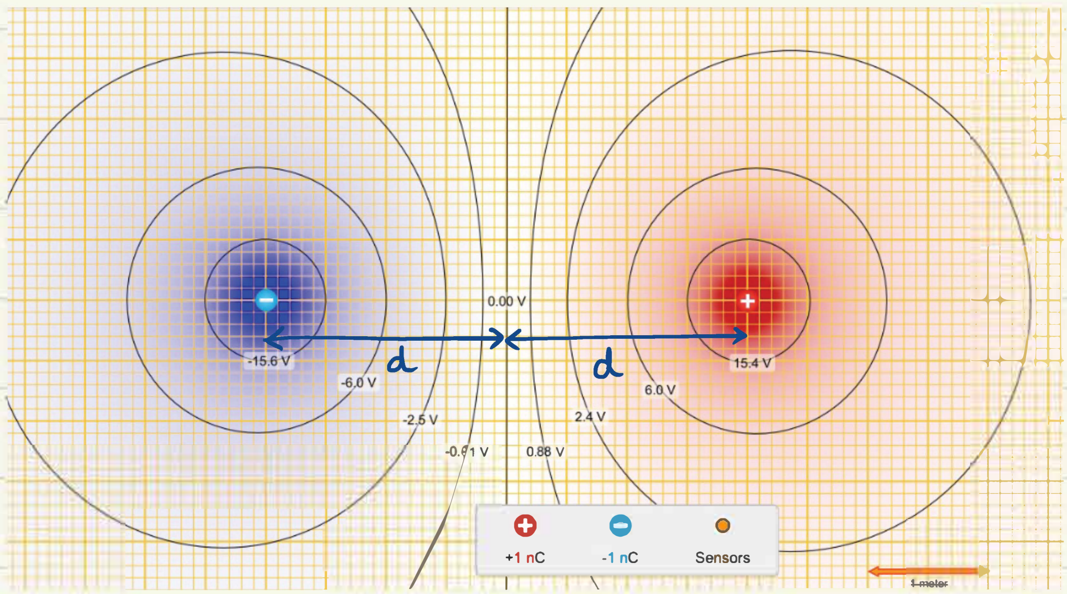
A point charge +q is held at a distance "d" in front of an infinite conducting plane (xy) that is grounded. Calculate the $\vec{E}$ over all space and the surface charge density induced on the plane.

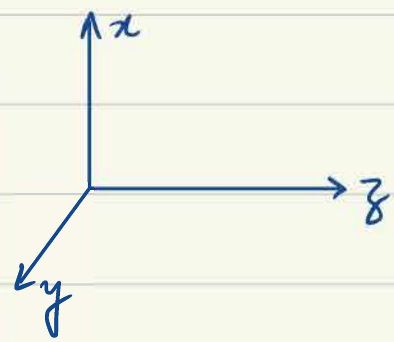
Boundary Conditions: ① $V=0$ for all $z=0$

② $V \to 0$ for $r \to \infty$ or $r \gg d$

Think of the conducting plane as a 0V Equipotential
Can you think of a charge config such that
an entire plane is V = 0V?

Remember the Equipotentials of a dipole?





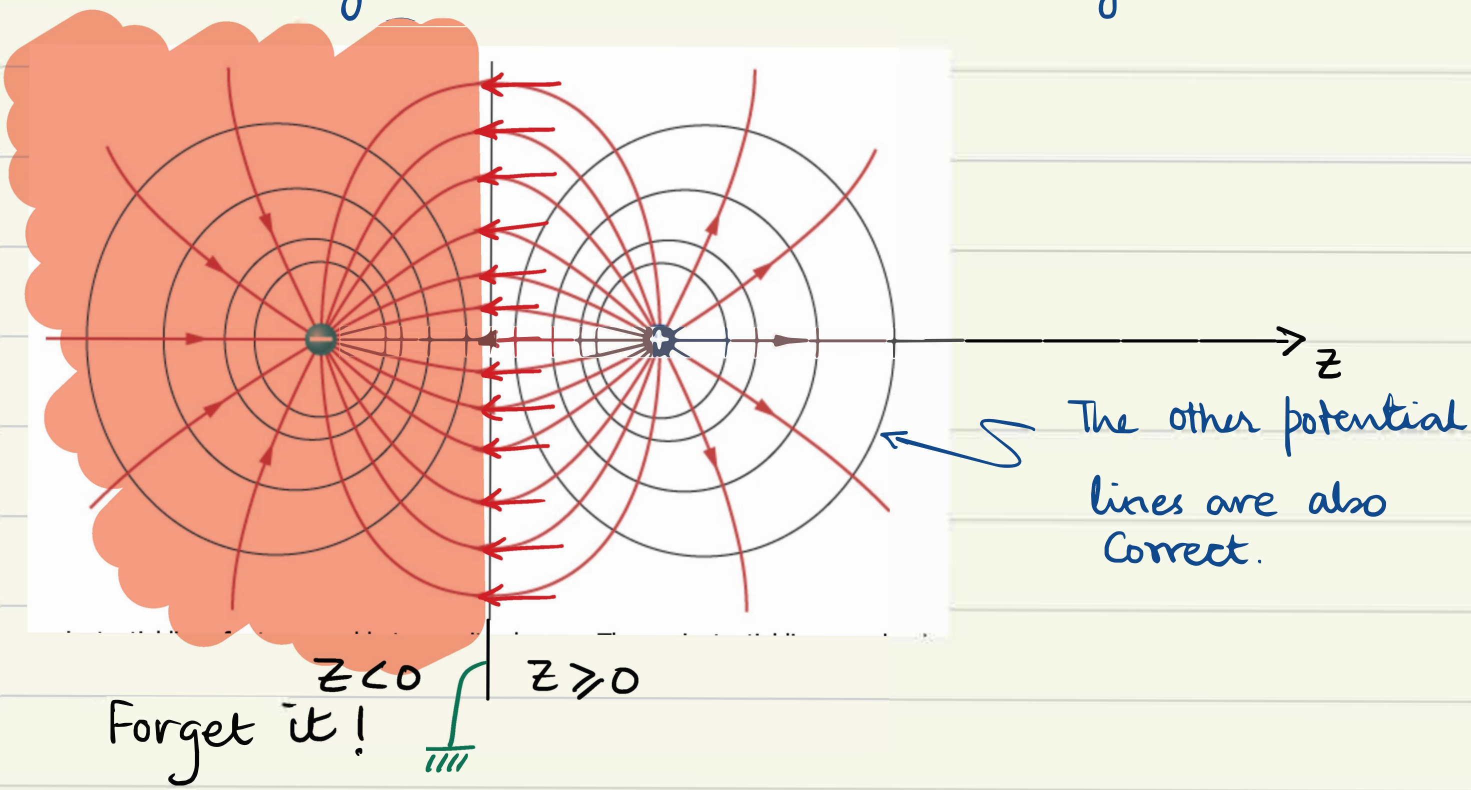
This plane is at V = 0V

it mimics the grounded conducting plane.

For this case $V(x,y,z) = \dfrac{q}{4\pi\epsilon_0}\left[\dfrac{1}{\left(x^2+y^2+(z-d)^2\right)^{1/2}} - \dfrac{1}{\left(x^2+y^2+(z+d)^2\right)^{1/2}}\right]$

Important: This potential satisfies the boundary conditions of our original problem. for $z \geq 0$!

$\Rightarrow$ The uniqueness theorem guarantees that if you have managed to find a valid solution then its the right solution and the only solution!



Field at the surface of Conducting plane is ⊥'r as expected and $E_\perp = \frac{\sigma}{\epsilon_0}\hat{k} \Rightarrow \sigma(x,y) = \epsilon_0 E_\perp(x,y,0)$ $\quad \left(E_\perp = E_z\Big|_{z=0}\right)$

$$\Rightarrow \sigma(x,y) = \epsilon_0\left(-\bar{\nabla}V\right)_{z=0} \text{ or } = \epsilon_0\left(-\frac{\partial V}{\partial z}\Big|_{z=0}\right)$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left[\frac{-q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

$$\frac{\sigma(x,y)}{\epsilon_0} = -\frac{\partial V}{\partial z}\Big|_{z=0} = \frac{-qd}{4\pi}\times\frac{2}{(x^2+y^2+d^2)^{3/2}} = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

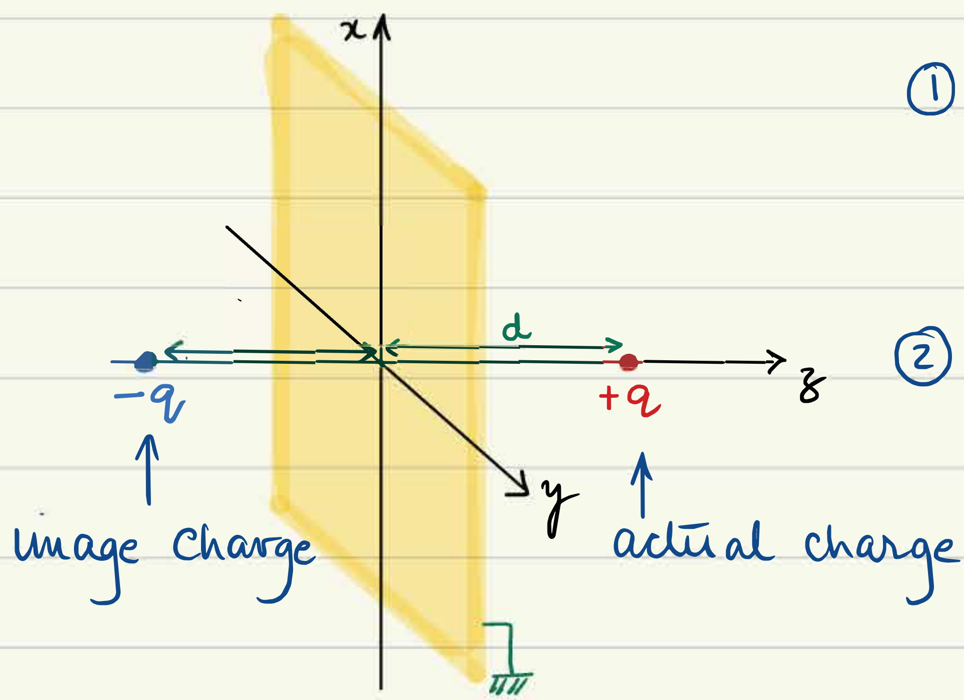
Note: ① Induced charge is $-ve$

② Cylindrically symmetric about z axis.

③ Total induced charge $Q = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} \frac{-qd\ \rho\, d\theta\, d\rho}{(\rho^2+d^2)^{3/2}}$

$$Q = \left.\frac{qd}{(\rho^2+d^2)^{1/2}}\right|_0^{\infty} = -q \qquad \rho^2 = x^2+y^2$$

Connecting with Reality – The Problem of Image Charge

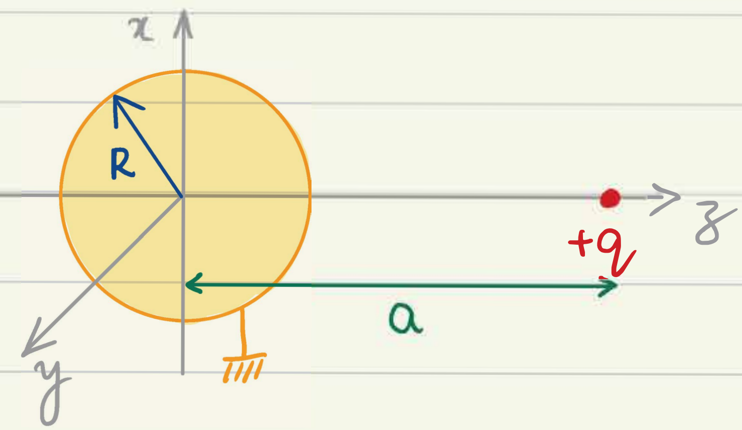


① Calculated $\bar{E}$, V and σ are valid in $z \geq 0$ only

② $\bar{E} = 0$ in $z < 0$ space

Faraday Cage.

Ex: Point charge in front of a conducting sphere.

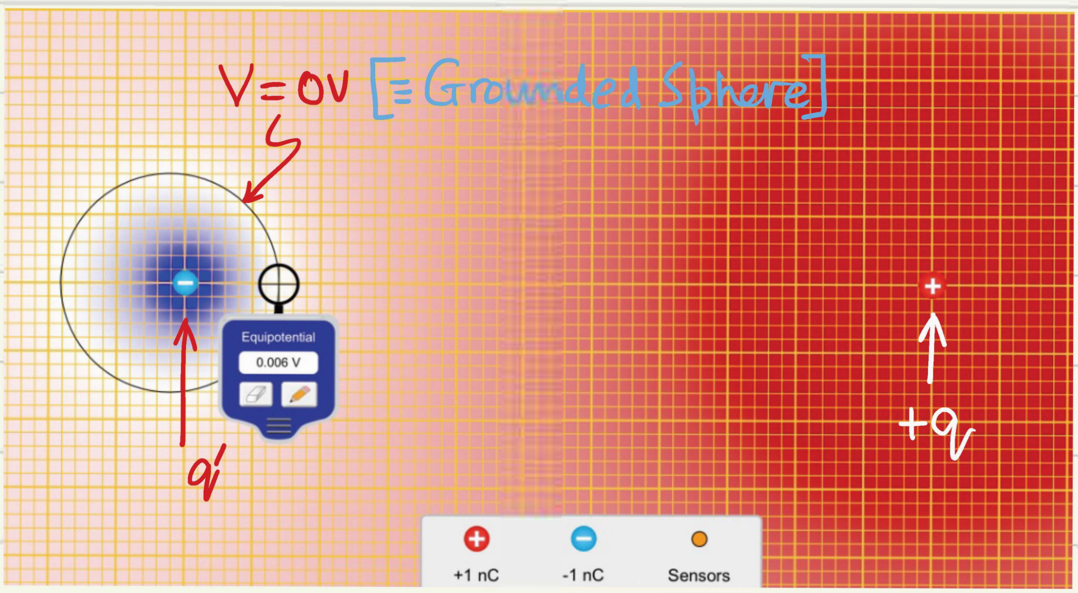


Boundary Conditions

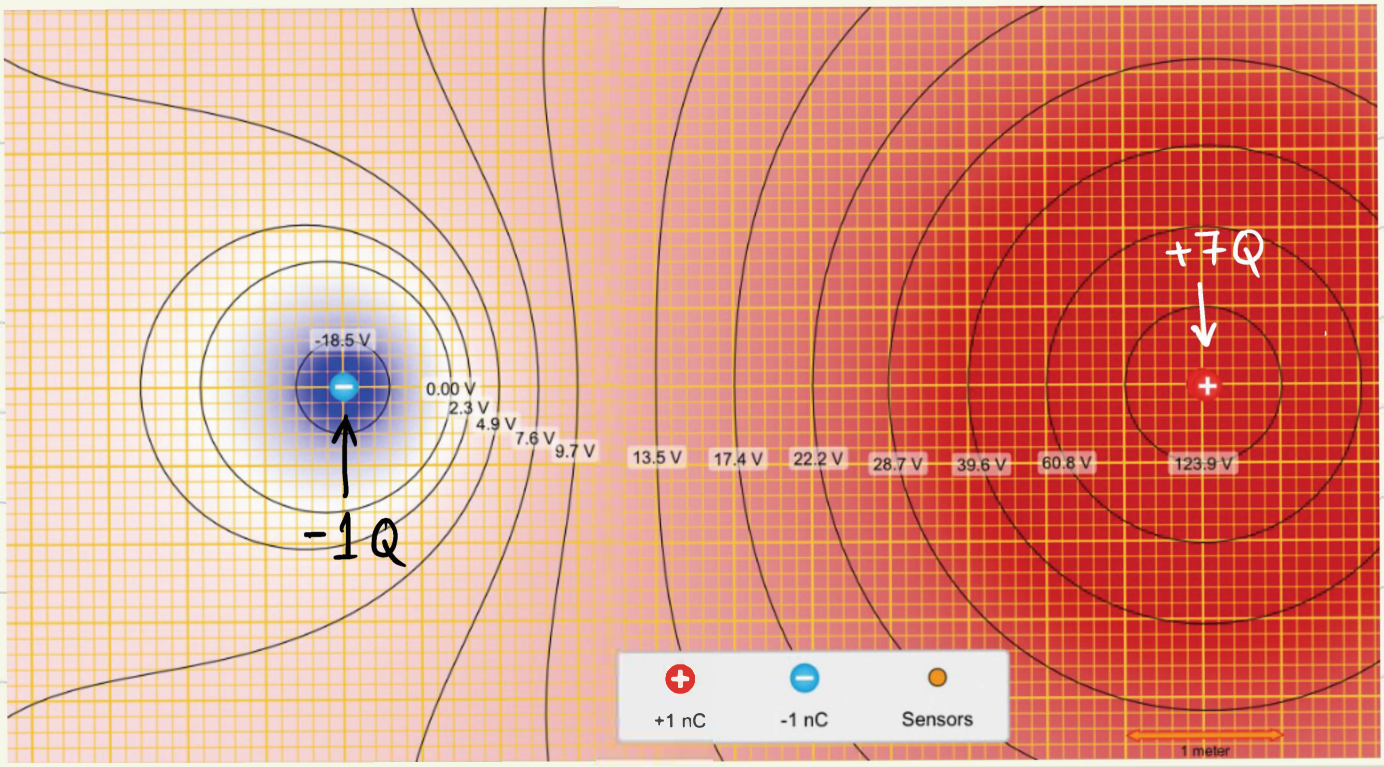
① $V = 0$ for $r = R$

② $V \to 0$ as $r \to \infty$

Can you think of an image charge configuration that results in a spherical equipotential of 0V?



$q' = -7q$ in this example



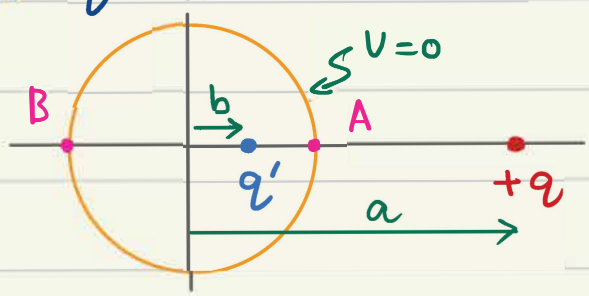
Plot of potential (colour) and Equipotential lines around 2 charges of magnitude $+q$ and $-q'$. $|q/q'| = 7$.

→ Image charge will lie within the sphere

→ Is of opposite sign

→ be off centre in the sphere

Assume that the image charge is q' at a distance b from centre of sphere.



$V = 0$ at all $r = R$

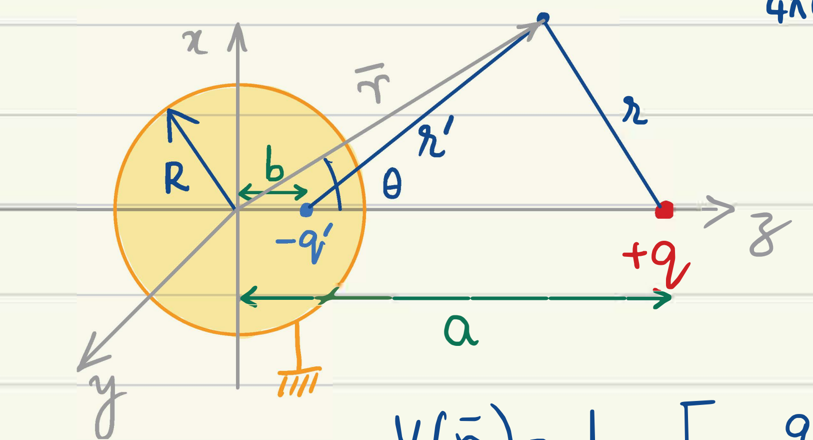
$$\Rightarrow \frac{q'}{R-b} + \frac{q}{a-R} = 0 \text{ at (A)} \quad \text{and} \quad \frac{q'}{R+b} + \frac{q}{a+R} = 0 \text{ at (B)}$$

$$\Rightarrow \frac{R+b}{R-b} = \frac{a+R}{a-R} \Rightarrow \frac{R}{b} = \frac{a}{R} \Rightarrow \boxed{b = R^2/a}$$

and $q' = -\frac{a}{R+a}\left(R + R^2/a\right) = -q\,\frac{R\,\cancel{(a+R)}}{\cancel{(R+a)}\,a}$

$\boxed{q' = -q\,\frac{R}{a}}$ and $\Rightarrow |q'| < |q|$

Potential: $V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z-b)^2}}\right]$



q : real Q
q' : image Q

The sphere divides the space into 2 parts
→ that of real Q ($r \geq R$)
→ that of image Q ($r < R$)

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(r^2+a^2-2ar\cos\theta)^{1/2}} + \frac{q'}{(r^2+b^2-2br\cos\theta)^{1/2}}\right]$$

$V \to 0$ as $r \to \infty$ (BC 2)

$V = ?$ at $r = R$

$$V(R) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(R^2+a^2-2aR\cos\theta)^{1/2}} + \frac{-Rq/a}{\left(R^2+\frac{R^4}{a^2}-2\frac{R^3}{a}\cos\theta\right)^{1/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(R^2+a^2-2aR\cos\theta)^{1/2}} + \frac{-R/a}{\frac{R}{a}(a^2+R^2-2aR\cos\theta)^{1/2}}\right]$$

$= 0$ for all θ and $r = R$ (BC 1)

To calculate $\sigma = \epsilon_0 E_\perp = \epsilon_0 E_r$ at the surface of sphere $(r = R)$

$$E_r = -\frac{\partial V}{\partial r} = \frac{-1}{4\pi\epsilon_0}\left[\frac{q\left(-\frac{1}{2}\right)(2r - 2a\cos\theta)}{(r^2 + a^2 - 2ar\cos\theta)^{3/2}} + \frac{q'\left(-\frac{1}{2}\right)(2r - 2b\cos\theta)}{(r^2 + b^2 - 2br\cos\theta)^{3/2}}\right]$$

$$-\left.\frac{\partial V}{\partial r}\right|_{r=R} = \frac{1}{4\pi\epsilon_0}\left[\frac{q(R - a\cos\theta)}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}} - \frac{Rq}{a}\frac{(R - R^2/a\cos\theta)}{\left(R^2 + \frac{R^4}{a^2} - 2\frac{R^2}{a}R\cos\theta\right)^{3/2}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{(R - a\cos\theta)}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}} - \frac{R^2(a - R\cos\theta)/a}{a\,\frac{R^3}{a^3}(a^2 + R^2 - 2Ra\cos\theta)^{3/2}}\right]$$

$$= \frac{q/4\pi\epsilon_0}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}}\left[(R - a\cos\theta) - \frac{(a - R\cos\theta)a}{R}\right]$$

$$\frac{\left[R^2 - aR\cancel{\cos\theta} - a^2 + aR\cancel{\cos\theta}\right]}{R}$$

$$\underset{(r=R)}{E_r} = \frac{-q}{4\pi\epsilon_0 R}\frac{(a^2 - R^2)}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}} = \sigma/\epsilon_0$$

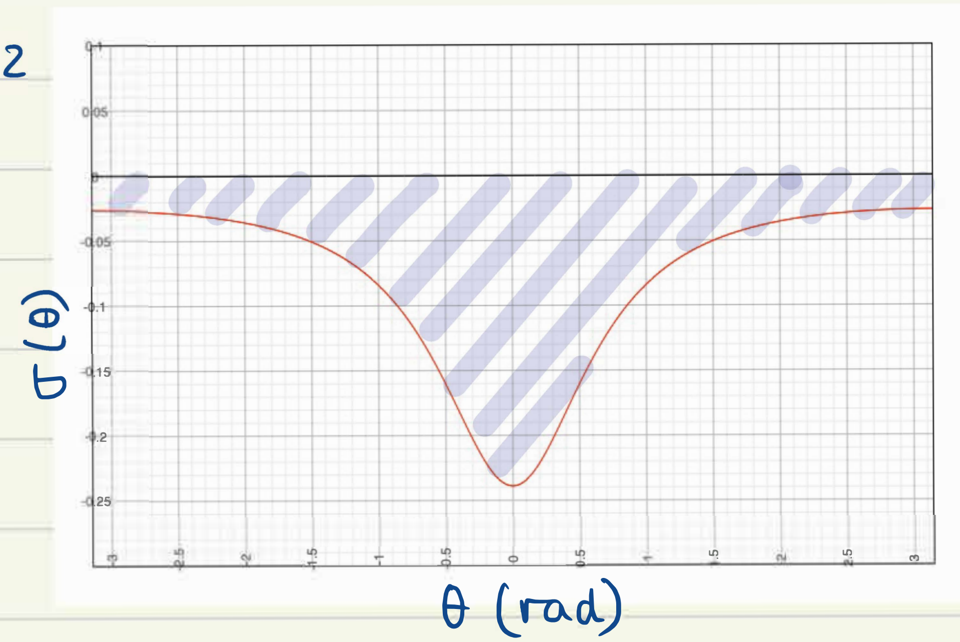
$\perp$'r component of $\bar{E}$ at surface of the sphere

$$\Rightarrow \sigma(\theta) = \frac{-q(a^2 - R^2)}{4\pi R(R^2 + a^2 - 2aR\cos\theta)^{3/2}} = \frac{-q\left(\frac{a^2}{R^2} - 1\right)}{4\pi R^2\left(1 + \frac{a^2}{R^2} - 2\frac{a}{R}\cos\theta\right)^{3/2}}$$

How does a plot of $\sigma(\theta)$ vs. θ look like?

$|\sigma(\theta)|$ is max for $\theta = 0^\circ$ and decreases with θ.

For $q = 1$, $R = 1$ & $a = 2$



Total induced charge

$$\int_{0}^{2\pi}\int_{0}^{\pi} \sigma(\theta) R^2 \sin\theta \, d\theta \, d\varphi$$

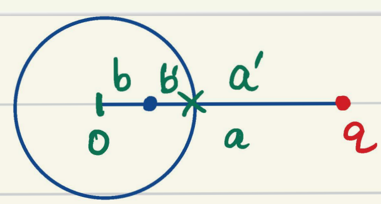
$\varphi \quad \theta$

$$= -\frac{qR}{a} = q'$$ (Homework – Complete the integral)

$\propto$ area under the curve above

Note: $\sigma(\theta)$ is never +ve at any θ. Why?

* What if sphere radius $R \to \infty$



$b = R^2/a$ and $q' = -qR/a$

Shift origin $(0,0,0) \to (0,0,R)$ $\quad a' = a - R \quad b' = R - b$

distance of q from surface of sphere

distance of q' from surface of sphere

$$\frac{a'}{R} + 1 = \frac{a}{R} \qquad \frac{b'}{R} = 1 - b/R$$

$$\Rightarrow q' = -q\left(1+\frac{a'}{R}\right)^{-1} \quad \text{and} \quad \frac{1}{b'} = \frac{1}{a'}\left(\frac{a'}{R}+1\right)$$

if $R \to \infty$ $\quad q' \to -q \quad$ and $\quad b' \to a'$

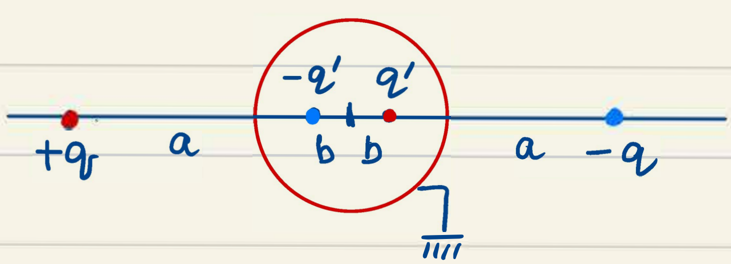
i.e. for a sphere of infinite radius $\equiv$ plane !!

* Say the sphere is not at $V=0$ but $V=V_0$.
Solve for the same problem with an additional point charge Q_0 at the origin such that $Q_0 = 4\pi\epsilon_0 R V_0$

↑ Centre of sphere. (in the image charge space)

* Say sphere is not grounded but is isolated and has a net charge Q.
Solve same problem but with charge $Q + qR/a$ at the centre (origin).

* Grounded Sphere in an uniform $\vec{E}$ field. $E_0\hat{k}$

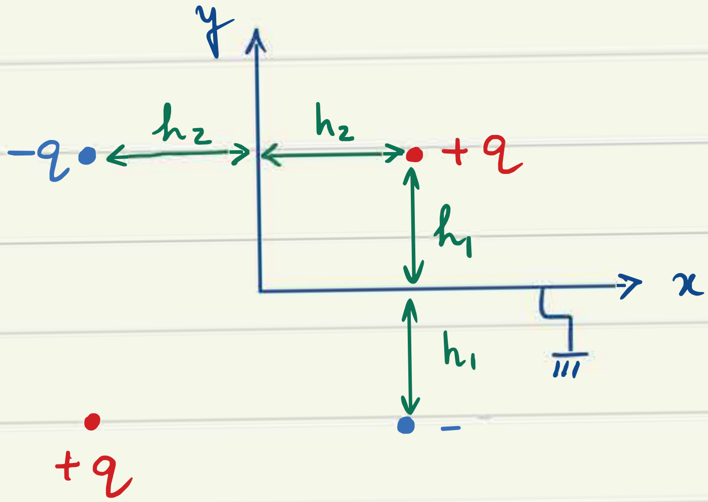


Potential at any point $r > R$ is given by that due to these 4 charges

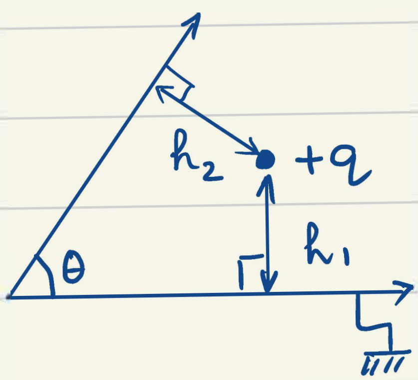
For the $+q$ $-q$ system the field in the region $r \ll a$ if they are far away from each other $(a \gg r)$ is

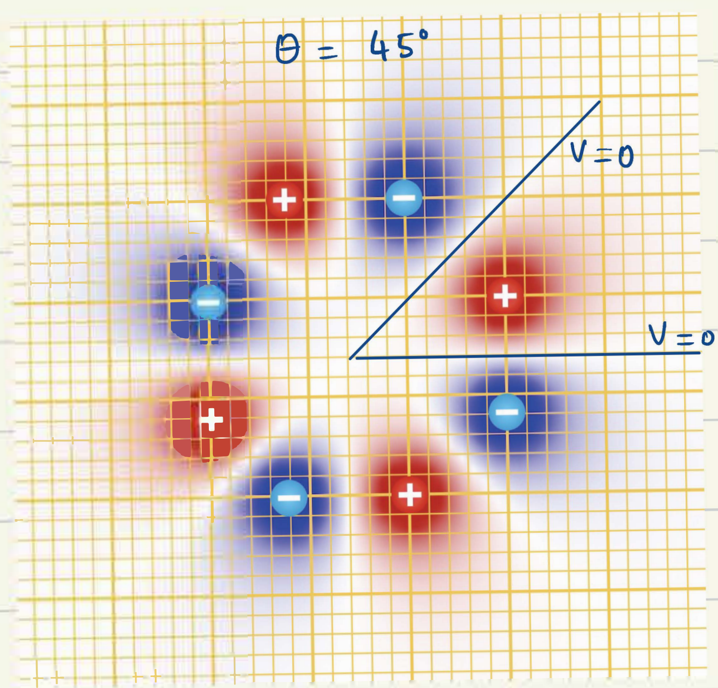
$$E_0\hat{k} = \frac{q}{2\pi\epsilon_0 a^2}\hat{k} \quad \text{(Prove it!)}$$

Ex: Calculate the system of image charges and their location. Calculate the force on the charge $+q$.



$$V(\bar{r}) = \sum_{i=1}^{4} \frac{1}{4\pi\epsilon_0} \frac{q_i}{r}$$





For arbitrary θ

Method of images works with finite number of image charges

if $\theta = \pi/n$

n: integer

Finally: Method of Images is a very useful techique.

A Trick that works in some cases not all.

① "Guess" an arrangement of charges (original charges + some extra charges) that replicate or satisfy the boundary conditions of the original problem.

② Extra charges — image charges must not be in the space where solution is wanted.

③ Calculate the $V(\bar{r})$ & $\bar{E}(\bar{r})$ in the actual space from ALL charges used → solution!!

The solutions are valid only in the real charge space and not in the image charge space.

Image charges can never be in the real charge space. That will change $\rho(\bar{r})$ of the original problem.

Laplace's Equation

Laplaces Eqn. appears in many contexts.

→ Heat Flow: $D\nabla^2 T = \dfrac{\partial T}{\partial t}$, in steady State $\dfrac{\partial T}{\partial t} = 0$

$$\Rightarrow \nabla^2 T = 0$$

→ Irrotational Flow of an incompressible, inviscid fluid.

$\nabla \times \bar{v} = 0$

$\Rightarrow \bar{v} = \nabla\varphi$

$$\underbrace{\rho = \text{const.} \qquad \eta = 0}_{\nabla\cdot\bar{v} = 0}$$

$$\Rightarrow \nabla^2\varphi = 0$$

→ Deflection of Elastic Membranes for small deformations

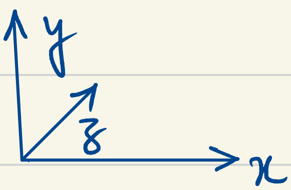
Drums / Bongo / Tabla

END

The material in the subsequent pages are not included in the syllabus of this Course and is extra material for those interested.

Application: Quadropole Lensing of Low Energy e⁻ beam

Beam of charge particles travelling along z direction



$$V(x,y) = \frac{x^2 - y^2}{a^2} V_{quad}$$

https://www.researchgate.net/deref/http%3A%2F%2Farxiv.org%2Fabs%2F1401.3951v1

What does the potential landscape along the xy plane and $\vec{E}$ field look like? (Homework)

Show that $V(x,y)$ focusses a +ve ion beam along x-direction and de-focusses along y-direction.

What about the solution of $\nabla^2 V = 0$ in general.

→ Solution is trivial in 1D: $V(x) = kx + m$

→ In 2D: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Take any Function of a complex var: $z = x + iy$

$F(z) = \underset{\uparrow}{u(x,y)} + i\underset{\uparrow}{W(x,y)}$ $\quad$ (Feynman Lec. Vol II Chap: 7)

real functions

It can be shown that u & W both satisfy Laplace's Eq for any $F(z)$.

Ex: $F(z) = z^2 = \underbrace{(x^2 - y^2)}_{u(x,y)} + 2i\underbrace{xy}_{W(x,y)}$

$u(x,y)$ is similar to the potential of the Quadropole Lens.

From the solutions we can "reverse engineer" the problem! — But works only in 2D.

$F(z) = \sqrt{z}, z^{2/3}, \ln z, 1/z$ → all correspond to actual V!

But even without complex variables: Direct Method.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \Rightarrow \underbrace{\frac{\partial^2 V}{\partial x^2} = -\frac{\partial^2 V}{\partial y^2}}_{\text{everywhere}} = k \text{ (constant)}$$

$$\frac{\partial^2 V}{\partial x^2} = k \Rightarrow \frac{\partial V}{\partial x} = kx + f(y) \Rightarrow V = \frac{kx^2}{2} + f(y)x + n$$

$$\frac{\partial^2 V}{\partial y^2} = -k \Rightarrow \frac{\partial V}{\partial y} = -ky + f'(x) \Rightarrow V = -\frac{ky^2}{2} + f'(x)y + n'$$

$$\Rightarrow V(x,y) = \frac{k}{2}(x^2 - y^2) + mxy + n$$

→ Separation of Variables (Griffiths Chap 3 sec 3.3)

Does not work in all cases. Works where the V and/or charges are specified at the bounding surface of a region where $\nabla^2 V = 0$.

1. Cartesian: $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$\because$ $x, y + z$ are arbitrary — above Eqn can be satisfied only if all 3 LHS terms are const. such that their sum $= 0$

$$\Rightarrow \left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= k_1^2 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= k_2^2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= k_3^2 \end{aligned}\right\} \quad k_1^2 + k_2^2 + k_3^2 = 0$$

Obviously all 3 k_i^2 cannot be +ve to yield the sum $= 0$

2. Spherical Polar Co-ordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}=0$$

if $V(r,\theta) \longrightarrow$ independent of φ (azimuthal symmetry)

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

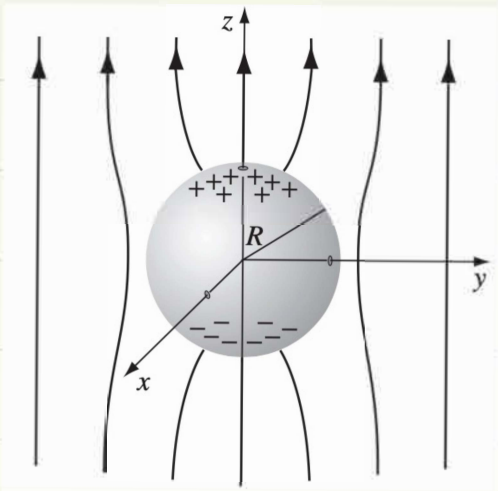
assume $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \Rightarrow R(r) = Ar^l + \frac{B}{r^{l+1}}$

and $\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$

$\Rightarrow \Theta(\theta) = P_l(\cos\theta) \leftarrow$ Legendre polynomials

$$\Rightarrow V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Application: Conducting Sphere in an uniform $\bar{E}$ field



$V(r, \theta) = ?$

Boundary Conditions:

(i) $V(r = R) = 0$

(ii) $V(r \to \infty) = -E_0 z = -E_0 r\cos\theta$

(i) $\Rightarrow$ $A_l R^l + \frac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = -A_l R^{2l+1}$

$$\Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

(ii) $\Rightarrow$ $\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$

$\Rightarrow l = 1 \Rightarrow A_1 = -E_0$

$A_i \, (i \neq 1) = 0$

$$\Rightarrow V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta$$

$$= \underbrace{-E_0 r\cos\theta}_{\text{External Field}} + \underbrace{\frac{E_0 R^3}{r^2} \cos\theta}_{\text{due to induced charge.}}$$

Induced Surface Charge density :-

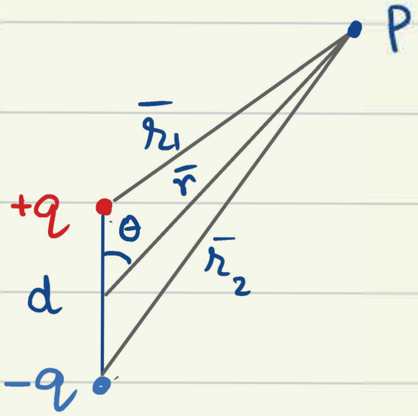
$$\frac{\sigma(\theta)}{\epsilon_0} = -\left.\frac{\partial V}{\partial r}\right|_R = E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\Big|_R = 3E_0\cos\theta$$

$$\Rightarrow \sigma(\theta) = 3\epsilon_0 E_0 \cos\theta$$

Chapter 4 Dipoles and Dielectrics: Electric fields in Matter

The Electric Dipole

PHYSICAL DIPOLE – 2 equal and opposite point charges separated by a fixed distance "d"



$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\left(r^2 + \frac{d^2}{4} - rd\cos\theta\right)^{1/2}} - \frac{q}{\left(r^2 + \frac{d^2}{4} + rd\cos\theta\right)^{1/2}}\right]$$

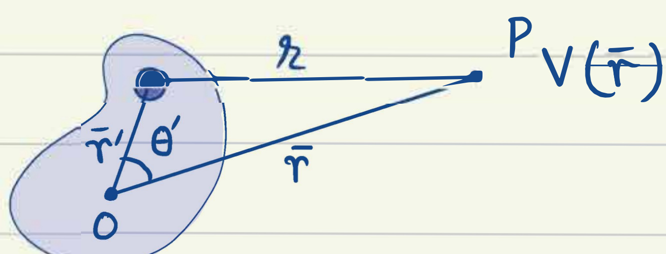
Assume $r \gg d \Rightarrow \left(r^2 + \frac{d^2}{4} \pm rd\cos\theta\right)^{-1/2} \simeq \frac{1}{r}\left(1 \mp \frac{d}{2r}\cos\theta\right)$

$$\Rightarrow V(\bar{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\left(1 + \frac{d}{2r}\cos\theta\right) - \left(1 - \frac{d}{2r}\cos\theta\right)\right]$$

$$= \frac{qd}{4\pi\epsilon_0 r^2}\cos\theta = \frac{\bar{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} \qquad \bar{p} = q\bar{d}$$

In fact for any arbitrary charge distribution $\rho(\bar{r})$

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_{VOL}\frac{\rho(\bar{r}')\,d\tau'}{\mathscr{r}}$$



$$\mathscr{r} = \left(r^2 + r'^2 - 2rr'\cos\theta'\right)^{1/2}$$

with a bit of algebra and binomial exp of $\frac{1}{r}$ we can show

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\overbrace{\int \rho(\bar{r}')\,d\tau'}^{\text{monopole term}} + \frac{1}{r^2}\overbrace{\int r'\rho(\bar{r}')\cos\theta'\,d\tau'}^{\text{dipole term}} + \frac{1}{r^3}\underbrace{\int r'^2\rho(\bar{r}')\left(\frac{3}{2}\cos\theta' - \frac{1}{2}\right)d\tau'}_{\text{qudropole term}} + \dots\right]$$

Together the above gives the Multipole expansion of an arbitrary charge distribution.

monopole dipole quadropole octopole

Monopole: $\frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\bar{r}')\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \leftarrow$ total charge

Dipole: $\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\rho(\bar{r}')\cos\theta'\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}\cdot\int \rho(\bar{r}')\,\bar{r}'\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot\hat{r}}{r^2}$

Dipole Potential

$$\boxed{V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot\hat{r}}{r^2}}$$

Dipole Moment

$$\boxed{\bar{p} = \int \rho(\bar{r}')\,\bar{r}'\,d\tau}$$

Note: Other than the monopole term ALL other terms are dependent on the choice of origin.

$\Rightarrow$ Even a single point charge has a non-zero dipole moment if the charge is not at the origin.

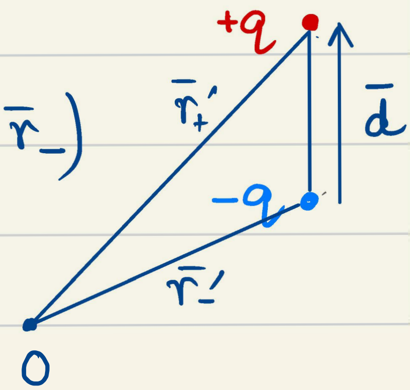
Exception: If the net charge of the distribution $Q = 0$ then the dipole moment $\bar{p}$ is independent of choice of the origin.

Caveat: The lowest non-vanishing moment is independent of the choice of origin — but not so for higher moments!

For a physical dipole of 2 point charges

$$\bar{p} = \sum_{i=1}^{2} q_i \bar{r}_i' = q\bar{r}_+' - q\bar{r}_-' = q(\bar{r}_+ - \bar{r}_-)$$

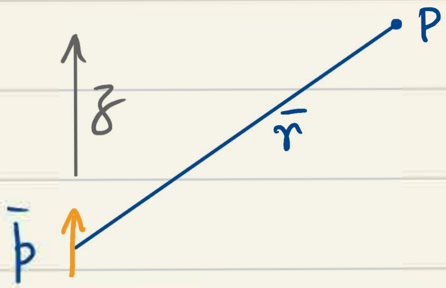
$$\bar{p} = q\vec{d} \quad (\bar{d} \text{ points from } -ve \text{ to } +ve)$$



Perfect Dipole : $d \to 0$ and $q \to \infty$ but $qd = p$

squeezed version of a physical dipole — finite

or we look at a physical dipole from far-far away



Assume $\bar{p}$ lies at the origin pointing along the $\bar{z}$ axis.

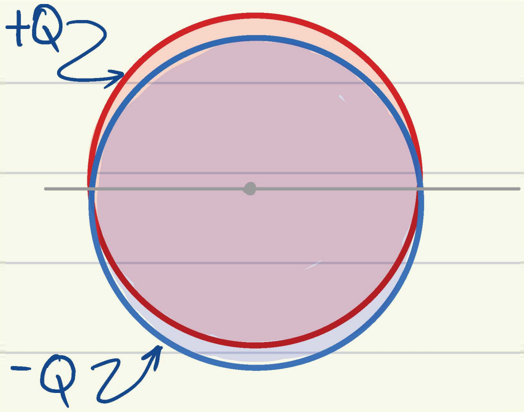
$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \frac{\bar{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{p\cos\theta}{r^2}$$

$$-\nabla V(\bar{r}) = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}\right) = \bar{E}(\bar{r})$$

$$\Rightarrow \bar{E}(\bar{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{2p\cos\theta}{r^3}\hat{r} + \frac{p\sin\theta}{r^3}\hat{\theta}\right)$$

$$\bar{E}(\bar{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\bar{p}\cdot\hat{r})\hat{r} - \bar{p}\right]$$ (Prove this!)

Ex: Say there are 2 spheres carrying equal and opposite uniform volume charge density. The centre of the spheres are at a distance d apart where $d \ll R$ (radii of the spheres). Calculate the $\bar{E}$ field over all space.



Centre of +Q is vertically displaced by $+d/2$

Centre of −Q is vertically displaced by $-d/2$

$\rho = 0$ in the overlap region.

$V(\bar{r})$ at any point outside the spheres $\equiv$ that due to 2 point charges at $(0,0,d/2)$ and $(0,0,-d/2)$ i.e. a physical dipole.

What about points inside the spheres, especially the overlap region. $r < R$.

Inside a sphere with uniform charge density

$\bar{E} = \dfrac{\rho \bar{r}}{3\epsilon_0}$ $\quad \bar{r}$ measured wrt sphere centre

If the centre shifts by $\frac{d}{2}$ and new position vector of point is $\bar{r}'$ then $\bar{r} = \bar{r}' - \dfrac{\bar{d}}{2}$

$\Rightarrow \bar{E}$ at $\bar{r}'$ is $\quad \bar{E} = \dfrac{\rho(\bar{r}' - \bar{d}/2)}{3\epsilon_0}$

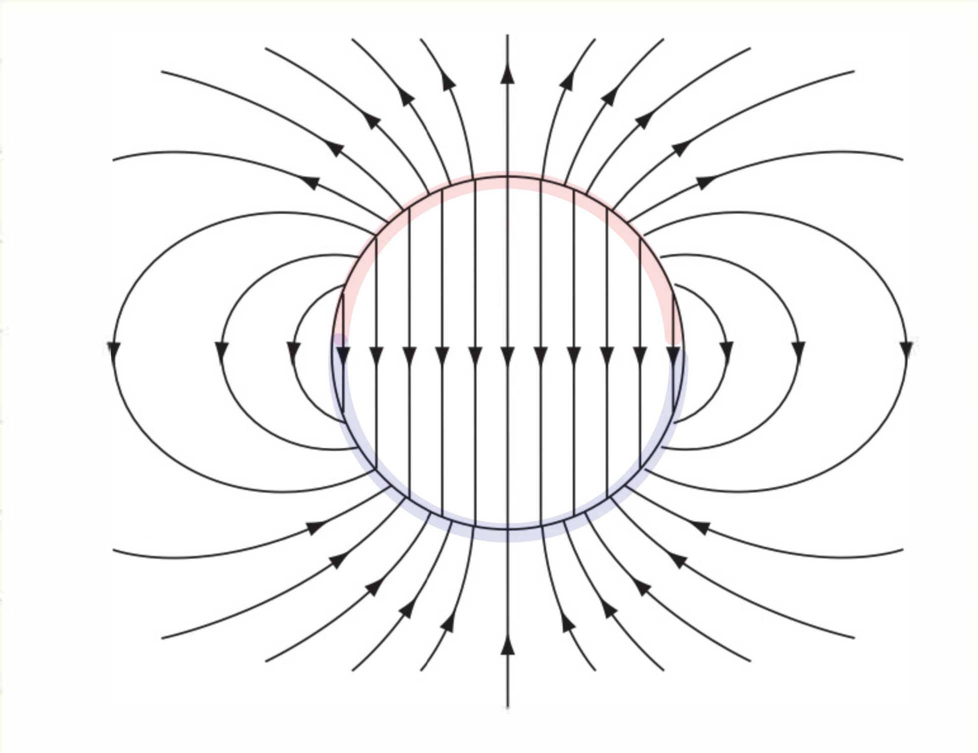
Similarly for the negative charge sphere

$$\bar{E}(\bar{r}') = -\frac{\rho(\bar{r}' + \bar{d}/2)}{3\epsilon_0}$$

$\Rightarrow$ Total $\bar{E}(\bar{r}') = -\dfrac{\rho \bar{d}}{3\epsilon_0} \hat{d}$

$\Rightarrow$ Total $\vec{E}$ is a constant throughout the overlap region and points along the direction of displacement.

How does the $\bar{E}$ field inside and outside look like?



Red and Blue shade indicate the un-overlapped +ve and −ve "surface charge".

Note: The $\bar{E}$ field is discontinuous at the region with surface charge.

Q: How do we calculate this "residual" surface charge density?

Ans: Calculate the discontinuity in the $E_\perp$ at $r = R$.

$$E_\perp(\text{above}) - E_\perp(\text{below}) = \sigma/\epsilon_0$$

Potential at any point outside the spheres: $(d \ll r \text{ or } R)$

$$V(\bar{r}) = \frac{q\,d}{4\pi\epsilon_0 r^2}\cos\theta \quad \text{and} \quad \bar{E}_\perp (\text{at } r = R) = -\left.\frac{\partial V}{\partial r}\right|_{r=R} \hat{r}$$

outside: $E_\perp(r = R) = \dfrac{q\,d\cos\theta}{2\pi\epsilon_0 R^3}$

inside: $E_\perp(r = R) = -\dfrac{\rho\,\bar{d}}{3\epsilon_0}\cdot\hat{r} = -\dfrac{\rho\,d\cos\theta}{3\epsilon_0}$

$$\Rightarrow E_\perp(\text{out}) - E_\perp(\text{in}) = \frac{q\,d\cos\theta}{2\pi\epsilon_0 R^3} + \frac{q}{\frac{4}{\cancel{3}}\pi R^3}\,\frac{d\cos\theta}{\cancel{3}\epsilon_0}$$

$$= \frac{1}{\epsilon_0}\left(\frac{3q\,d\cos\theta}{4\pi R^3}\right)$$

$$= \left(\frac{\rho\,d\cos\theta}{\epsilon_0}\right) = \sigma/\epsilon_0$$

$\Rightarrow \sigma(\theta) = \rho\,d\cos\theta$

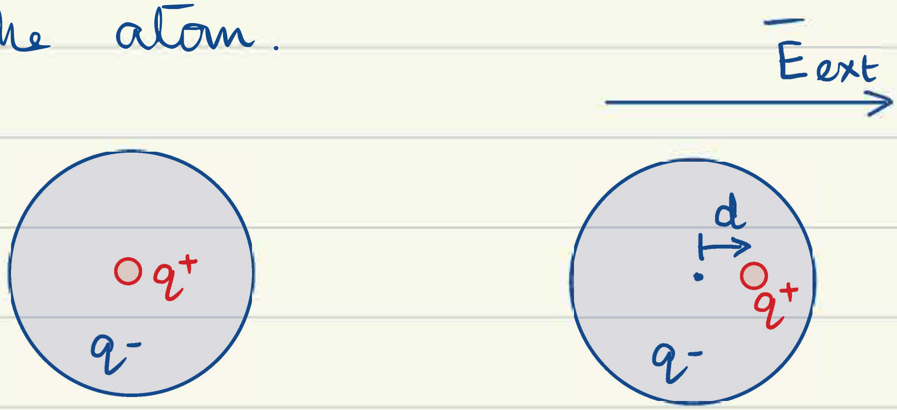
this is dimensionally correct!

σ is +ve for $\theta = 0 \to \pi/2$

σ is −ve for $\theta = \pi/2 \to \pi$

$\sigma = 0$ at $\theta = \pi/2$

And this is what is going to happen if you impose an external $\bar{E}$ field onto an atom. atleast in the case of zeroth order model of the atom.



dipole moment

$p = 0$

induced dipole moment

$p = qd$

At Equilibrium Force on $+q$ is zero.

$$\Rightarrow q E_{ext} = q \underbrace{\frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}}_{}$$

Field inside a spherical charge density at a distance d from origin.

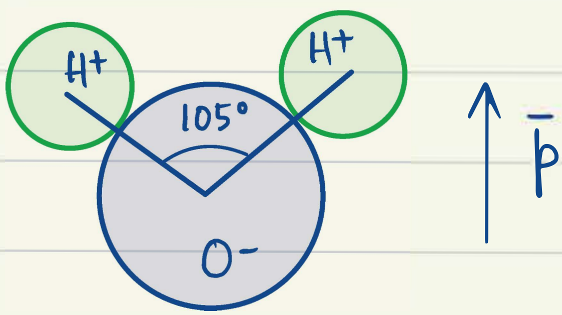
$$\Rightarrow qd = (4\pi\epsilon_0 a^3) E_{ext} = p \text{ (induced)}$$

We assumed that shape of the e^- cloud remains spherical which is not true.

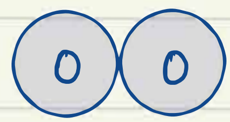
$$\bar{p} = (4\pi\epsilon_0 a^3)\bar{E} \Rightarrow \text{Atomic polarizability}$$

$$\alpha = 4\pi\epsilon_0 a^3$$

Molecules may be polar or non-polar.

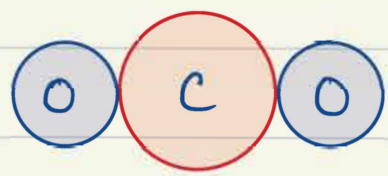


H_2O



O_2 molecule

$\bar{p} = 0$



CO_2 molecule

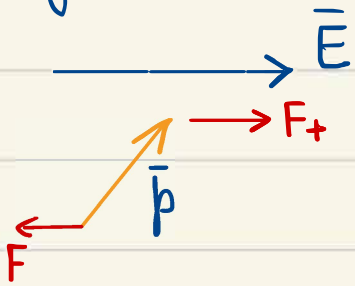
$\bar{p}$

In an external $\bar{E}$ field molecules and atoms in solids / liquids / gasses may

STRETCH → polarization

ROTATE → alignment

So how do existing dipoles interact with an external field?



net force on the dipole in uniform $\bar{E}$

$\because$ $+q\bar{E}$ acts on +ve end

$-q\bar{E}$ acts on the −ve end

Net Force $= q\bar{E} - q\bar{E} = 0$

But there is a non-zero torque acting on the dipole. (Rem: $\bar{p}$ is constant)

Assuming the dipole lies at the origin.

$$(\text{Torque})\ \bar{N} = (\bar{r}_+ \times q\bar{E}) + (\bar{r}_- \times q\bar{E}) = q\bar{d} \times \bar{E} = \bar{p} \times \bar{E}$$

$$\boxed{\bar{N} = \bar{p} \times \bar{E}} \qquad |\bar{N}| = pE\sin\theta$$

The torque acts such that the dipole is forced to align along $\bar{E}$

However if $\bar{E}$ is not uniform the forces at the two ends of the dipole are not equal and opposite.

$$\Rightarrow \quad \bar{F} = \bar{F}_+ + \bar{F}_- = q\underbrace{(\bar{E}_+ - \bar{E}_-)}_{\Delta \bar{E}}$$

$$\Delta\bar{E} = \Delta E_x \hat{i} + \Delta E_y \hat{j} + \Delta E_z \hat{k}$$

$$= (\bar{\nabla} E_x . \bar{d})\hat{i} + (\bar{\nabla} E_y . \bar{d})\hat{j} + (\bar{\nabla} E_z . \bar{d})\hat{k}$$

$$\Rightarrow \Delta\bar{E} = (\bar{d}.\bar{\nabla})\bar{E}$$

$$\Rightarrow \bar{F} = q\Delta\bar{E} = q(\bar{d}.\bar{\nabla})\bar{E}$$

$$\boxed{\bar{F} = (\bar{p}.\bar{\nabla})\bar{E}}$$

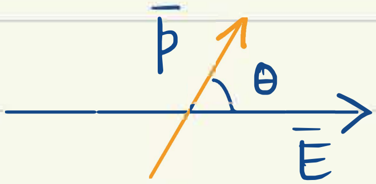
$$\bar{F} = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z}\right)\left(E_x \hat{i} + E_y \hat{j} + E_z \hat{k}\right)$$

$$F_x = \left(p_x \frac{\partial E_x}{\partial x} + p_y \frac{\partial E_x}{\partial y} + p_z \frac{\partial E_x}{\partial z}\right) \; ; \; F_y = ? \; ; \; F_z = ?$$

This finds application in optical traps and tweezers.

Energy of a dipole in an $\bar{E}$ field.

A dipole held at an arbitrary angle with the local field has potential energy.



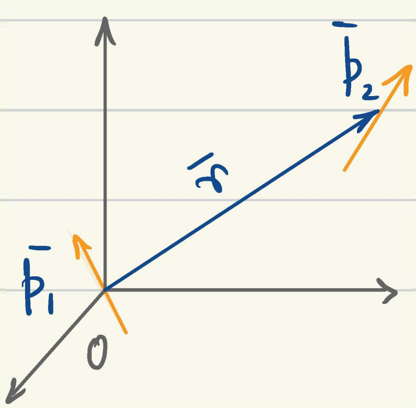
$$PE = +q\,V(\bar{r}+\bar{d}) - q\,V(\bar{r})$$

$$= q\left\{V(\bar{r}+\bar{d}) - V(\bar{r})\right\}$$

$$= -q\int_{\bar{r}}^{\bar{r}+\bar{d}} \vec{E}\cdot\bar{dl} \quad = \quad -q\,\vec{E}\cdot\bar{d}$$

Potential Energy $U = -\bar{p}\cdot\bar{E}$ $\qquad$ $\bar{p} = q\bar{d}$

Ex: Calculate the P.E. of a system of 2 dipoles separated by $\bar{r}$. $\bar{p}_1$ lies at the origin and $\bar{p}_2$ has position $\bar{r}$



$\bar{E}$ field due to $\bar{p}_1$ at $\bar{r}$ $(\bar{p}_2)$

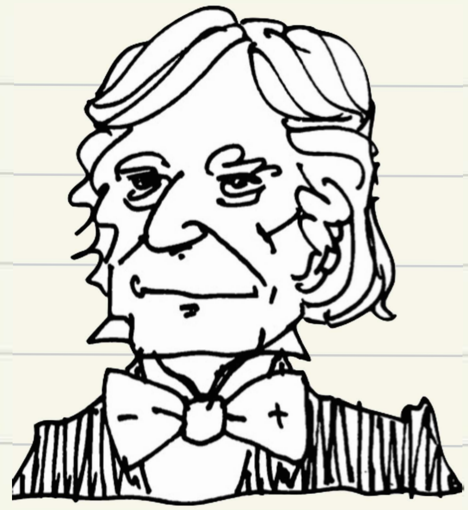
$$\bar{E}_1(\bar{r}) = \frac{1}{4\pi\epsilon_0 r^3}\left(3(\bar{p}_1\cdot\hat{r})\hat{r} - \bar{p}_1\right)$$

$\Rightarrow$ P.E of $\bar{p}_2$ due to the external field $\bar{E}_1$

$$P.E. = -\bar{p}_2 \cdot \bar{E}_1$$

$$-\bar{p}_2 \cdot \bar{E}_1 = \frac{-1}{4\pi\epsilon_0 r^3}\left[3(\bar{p}_1 \cdot \hat{r})(\bar{p}_2 \cdot \hat{r}) - \bar{p}_1 \cdot \bar{p}_2\right]$$

Dielectrics

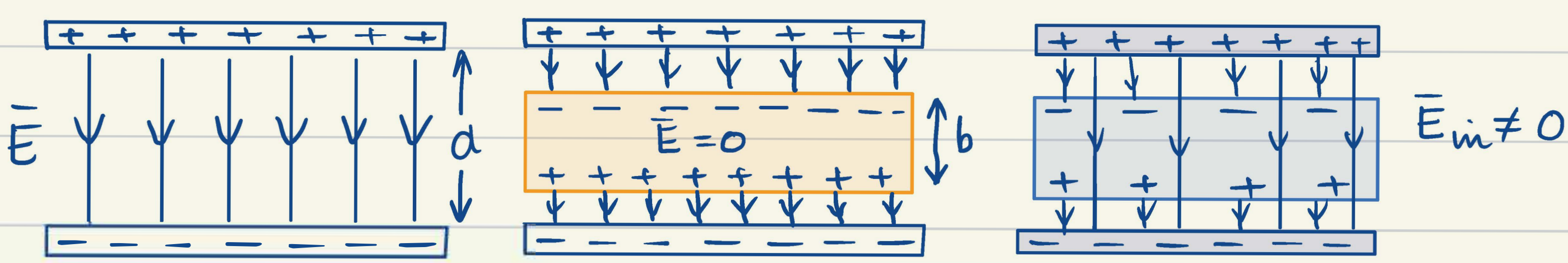


FARADAY
(1791·1867).

Michael Faraday observed that the charge holding capacity of capacitors i.e. capacitance increased when insulators were inserted between the parallel plates.

These materials are called Dielectrics and are characterized by a constant k known as the dielectric constant.

Say we have a // plate capacitor. What happens if we introduced a perfect conductor/dielectric in the space?



$$C = \frac{\epsilon_0 A}{d}$$

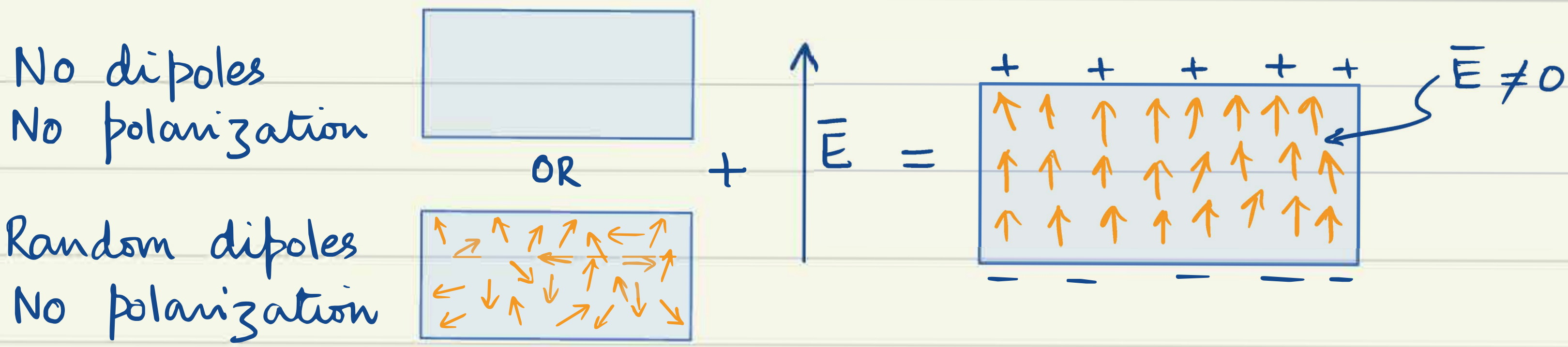
$$C = \frac{\epsilon_0 A}{(d-b)}$$

Full cancellation of $\bar{E}$ inside conductor

$$C = ?$$

Partial cancellation of $\bar{E}$ inside dielectric

⇒ A diectric in an external field will get polarized. Local dipole moments will be induced or existing polar components will be aligned (even partially)

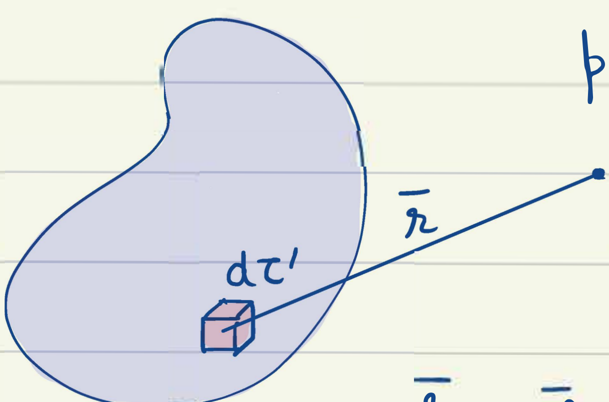


Streching + rotation ⟶ Polarization

$$\boxed{\bar{P} = \text{dipole moment/unit volume}}$$

What is the field produced by a polarized system?

Note : The system may not be uniformly polarized & there is no $\bar{E}_{ext}$



$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_{VOL} \frac{\bar{P}(\bar{r}') \cdot \hat{\imath}}{\imath^2} \, d\tau'$$

$\bar{\imath} = \bar{r} - \bar{r}'$ $\qquad$ $\bar{r}$: position of field

$\bar{r}'$: position of source

Also valid if $\bar{r}$ lies inside the system G 4.2.3

$V(\bar{r})$ can also be written as;

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_V \bar{P} \cdot \nabla' \left(\frac{1}{r}\right) d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \left[\int_V \nabla' \cdot \left(\frac{\bar{P}}{r}\right) d\tau' - \int_V \frac{(\nabla' \cdot \bar{P})}{r} d\tau' \right]$$

$\downarrow$ apply divergence theorem

$$\Rightarrow \quad V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \left[\oint_S \frac{\bar{P}}{r} \cdot d\bar{a}' + \int_V \frac{-(\nabla' \cdot \bar{P})}{r} d\tau' \right]$$

potential of a surface charge density

potential due to a volume charge density

$$\boxed{\sigma_b = \bar{P} \cdot \hat{n}}$$

$$\boxed{\rho_b = -\bar{\nabla}' \cdot \bar{P}}$$

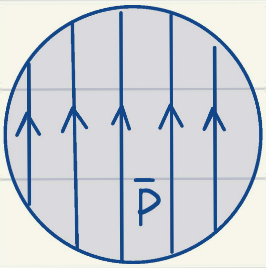
$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \oint \frac{\sigma_b}{r} da' + \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_b}{r} d\tau'$$

The subscript "b" denotes that these charges are bound.

Note: If polarization inside the system is uniform then $\nabla' \cdot \bar{P} = 0 \Rightarrow$ no bound charge density

$\Rightarrow$ all the induced charges will lie at the surface of the system.

Ex: Uniformly polarized Sphere



$$\Rightarrow \quad \sigma_b = \bar{P} \cdot \hat{n} = \bar{P} \cdot \hat{r} = P \cos\theta$$

$$\Rightarrow \quad \rho_b = -\nabla' \cdot \bar{P} = 0$$

Effects of Polarization is to produce bound charges.

1. Induce surface charge σ_b
2. Produce volume charge ρ_b

$\Rightarrow$ Total charge density $\rho = \rho_b + \rho_f$

free charges

Gauss' Law: $\bar{\nabla}.\bar{E} = \rho/\epsilon_0$

$$\Rightarrow \bar{\nabla}.\bar{E} = \frac{\rho_b + \rho_f}{\epsilon_0} = \frac{\rho_b}{\epsilon_0} + \frac{\rho_f}{\epsilon_0}$$

$$= \frac{-\bar{\nabla}.\bar{P}}{\epsilon_0} + \frac{\rho_f}{\epsilon_0}$$

$$\Rightarrow \bar{\nabla}.(\epsilon_0\bar{E}) = -\bar{\nabla}.\bar{P} + \rho_f$$

$$\Rightarrow \bar{\nabla}.(\epsilon_0\bar{E} + \bar{P}) = \rho_f$$

Term in ( ) on the LHS

$\boxed{\epsilon_0\bar{E} + \bar{P} = \bar{D}}$ Electric Displacement vector

In many ways $\bar{D}$ behaves like $\bar{E}$, but not in ALL ways

$\bar{D}$ satisfies a form of Gauss' Law

$$\boxed{\bar{\nabla}.\bar{D} = \rho_f} \Rightarrow \boxed{\oint \bar{D}.d\bar{a} = Q_f}$$

$$\bar{\nabla}.\bar{E} = \frac{\rho_b + \rho_f}{\epsilon_0} \longrightarrow \oint \bar{E}.d\bar{a} = (Q_b + Q_f)/\epsilon_0$$

In problems involving dielectric media and free charges (ρ_f) $\rightarrow \bar{P}, \sigma_b, \rho_b$ are not known apriori $\Rightarrow$ starting from $\bar{\nabla}.\bar{E} = \rho/\epsilon_0$ is not possible. But

But since ρ_f is known we can solve $\bar{\nabla}.\bar{D} = \rho_f$ and obtain $\bar{D}$.

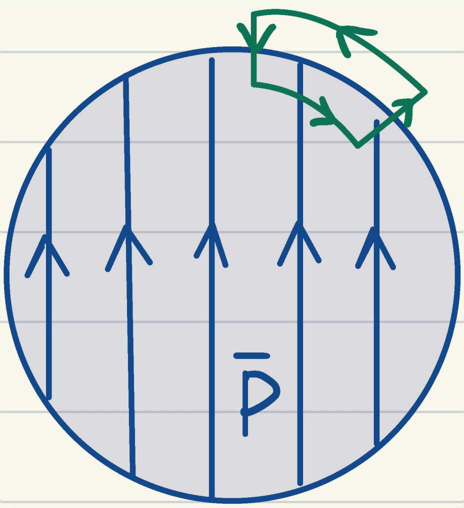
However, there are important differences;

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P}$$

$$\Rightarrow \bar{\nabla} \times \bar{D} = \epsilon_0 \bar{\nabla} \times \bar{E} + \bar{\nabla} \times \bar{P} \quad (\bar{\nabla} \times \bar{E} = 0 \text{ always})$$

$$\Rightarrow \bar{\nabla} \times \bar{D} = \bar{\nabla} \times \bar{P} \neq 0 \quad \text{and} \quad \bar{D} \neq \underset{\substack{\uparrow \\ \text{scalar Function}}}{\nabla W}$$

$\bar{D}$ is not given by the gradient of a scalar



Outside the sphere $\bar{D} = \epsilon_0 \bar{E}$

Inside the sphere $\bar{D} = \epsilon_0 \bar{E} + \bar{P}$

Compute $\oint \bar{D} . \bar{dl}$ along the Green path.

$$\oint \bar{D} . \bar{dl} = \int_{in} (\epsilon_0 \bar{E} + \bar{P}) . \bar{dl} + \int_{out} \epsilon_0 \bar{E} . \bar{dl}$$

$$\Rightarrow \int_{in} \epsilon_0 \bar{E} . \bar{dl} + \int_{out} \epsilon_0 \bar{E} . \bar{dl} + \int_{in} \bar{P} . \bar{dl} = \int_S (\nabla \times \bar{D}) . \bar{da}$$

$$\underset{\substack{\| \\ 0}}{\epsilon_0 \oint \bar{E} . \bar{dl}} + \underset{\substack{\nparallel \\ 0}}{\oint \bar{P} . \bar{dl}} = \oint \bar{D} . \bar{dl} = \int (\bar{\nabla} \times \bar{D}) . \bar{da} \ (\text{Stokes' Th})$$

$$\Rightarrow \int (\bar{\nabla} \times \bar{D}) . \bar{da} = \oint \bar{P} . \bar{dl} \neq 0$$

$$\Rightarrow \bar{\nabla} \times \bar{D} \neq 0 \quad \Rightarrow \quad \bar{D} \neq \nabla W$$

(Effect of external $\bar{E}$ field on a Dielectric)

Linear Dielectrics

Dielectrics in which the induced polarization by an external Field $\bar{E}_0$ is ;

$$\boxed{\bar{P} = \epsilon_0 \chi_0 \bar{E}}$$

χ_0 : electric susceptibility, $\bar{E}$ is not too strong!
$\chi_0 = 0$ for vacuum

Note: Here $\bar{P}$ is the local dipole mom./unit volume and $\bar{E}$ is the total local field not $\bar{E}_0$ only.

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} \Rightarrow \boxed{\bar{D} = \epsilon_0 (1+\chi_0)\, \bar{E} = \epsilon \bar{E}} \qquad \boxed{\bar{D} = \left(\frac{1+\chi_0}{\chi_0}\right) \bar{P}}$$

ϵ : permittivity of media

$$\boxed{\epsilon = \epsilon_0 (1+\chi_0)} \quad \boxed{\frac{\epsilon}{\epsilon_0} = \epsilon_r = 1+\chi_0}$$

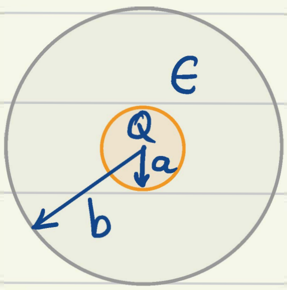
relative permittivity
dielectric const.

$\epsilon_r = 1$ for vacuum and $\epsilon_r > 1$ for material $\Rightarrow$ $\epsilon > \epsilon_0$

ϵ_r of some materials

Vacuum	1
Air	1.000536
Water vapour	1.00589
Silicon	11
Mica	5
Benzene	2.3
Water	80
Ice	104
Hafnium Oxide	60
KTaNb03	34,000
CaCuTiO	100,000

Ex: A metal sphere of radius a carries a charge Q. A diectric of permittivity ϵ surrounds the sphere as shown in the figure. Calculate the potential difference between the centre of sphere and ∞.



$$\text{Potential difference} = -\int_{\infty}^{0} \bar{E}\cdot d\bar{l}$$

Region I : Inside sphere $\quad \bar{E} = 0$

Region $\underline{\text{II}}$: Iaside dielectric

Gauss law $\quad \oint \bar{D}\cdot da = Q$ free charge on the conducting sphere

$$\because \bar{D} = \epsilon\bar{E} \Rightarrow \oint \epsilon\bar{E}\cdot d\bar{a} = Q \quad \Rightarrow \quad \bar{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r} = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2}\hat{r}$$

Note: The $\bar{E}$ field in the region of the dielectric is smaller than what it would have been in vacuum by the factor ϵ_r

Region $\underline{\text{III}}$: Outside Dielectric

Again $\oint \bar{D}\cdot d\bar{a} = Q$, but here $\bar{D} = \epsilon_0\bar{E}$

$$\Rightarrow \bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

Note: We start with $\bar{D}$ and not $\bar{E}$ since we know only the free charge density and not the total charge densities in the system (conductor + dielectric)

$$\Rightarrow V(0)-V(\infty) = -\int_{\infty}^{0} \bar{E}\cdot d\bar{l} = \underbrace{-\int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{b}^{a} \frac{Q}{4\pi\epsilon r^2} dr}_{\swarrow} - \int_{a}^{0} \underset{\underset{0}{\parallel}}{\bar{E}\cdot d\bar{r}}$$

[inside conductor

Potential Difference $= \dfrac{Q}{4\pi}\left(\dfrac{1}{\epsilon_0 b} + \dfrac{1}{\epsilon a} - \dfrac{1}{\epsilon b}\right)$

Inside the dielectric $\bar{P} = \epsilon_0 \chi_0 \bar{E}$ (Region II)

$$\Rightarrow \bar{P} = \epsilon_0 \chi_0 \frac{Q}{4\pi\epsilon r^2}\hat{r} = \frac{\chi_0}{1+\chi_0}\frac{Q}{4\pi r^2}\hat{r}$$

We can calculate the Bound Charges;

volume charge density $\rho_b = -\bar{\nabla}\cdot\bar{P} = -\dfrac{\chi_0 Q}{(1+\chi_0)4\pi}\nabla\cdot\left(\dfrac{\hat{r}}{r^2}\right) = 0$

and

Surface charge density $\sigma_b = \bar{P}\cdot\hat{n}$

Outer surface, $r=b$; $\hat{n}=\hat{r} \Rightarrow \bar{P}\cdot\hat{r}\Big|_{r=b} = \dfrac{\epsilon_0\chi_0 Q}{4\pi\epsilon b^2}$

Inner surface, $r=a$; $\hat{n}=-\hat{r} \Rightarrow \bar{P}\cdot-\hat{r}\Big|_{r=a} = -\dfrac{\epsilon_0\chi_0 Q}{4\pi\epsilon a^2}$

Boundary Conditions

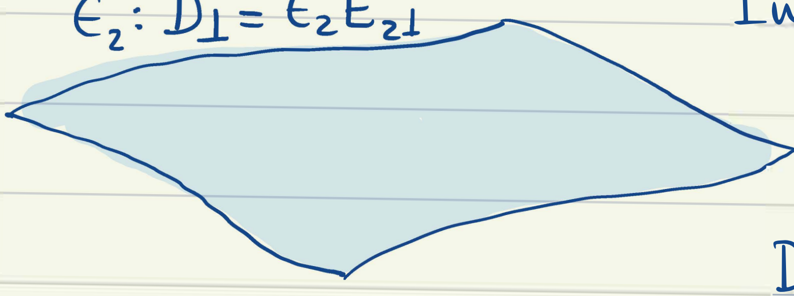
$$V_{above} = V_{below}$$

$E_\perp(\text{above}) - E_\perp(\text{below}) = \sigma/\epsilon_0$	$E_{//}(\text{above}) - E_{//}(\text{below}) = 0$ $(\nabla \times \bar{E} = 0)$
$D_\perp(\text{above}) - D_\perp(\text{below}) = \sigma_f$	$D_{//}(\text{above}) - D_{//}(\text{below})$ $= P_{//}(\text{above}) - P_{//}(\text{below})$

$(\nabla \times \bar{D} \neq 0)$

For Linear Dielectrics: $(\bar{D} = \epsilon \bar{E})$

$\epsilon_2 : D_\perp = \epsilon_2 E_{2\perp}$



$\epsilon_1 : D_\perp = \epsilon_1 E_{1\perp}$

Interface between 2 dielectrics with no free charges $\rho_f = 0$

$$D_\perp(\text{above}) - D_\perp(\text{below}) = \sigma_f$$

$$\Rightarrow \boxed{\epsilon_2 E_{2\perp} - \epsilon_1 E_{1\perp} = \sigma_f}$$

$$\boxed{E_{2//} = E_{1//}}$$

Bound Charge Density

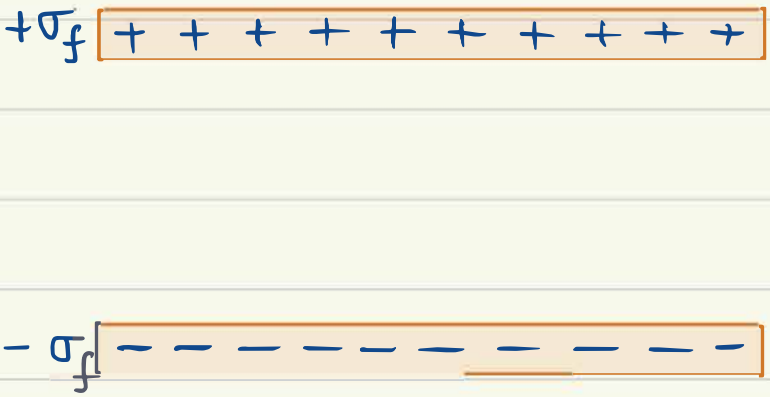
$$\rho_b = -\bar{\nabla} . \bar{P}$$

$$= -\bar{\nabla} . \bar{D} \frac{\chi_0}{1+\chi_0}$$

$$\boxed{\rho_b = -\frac{\chi_0}{1+\chi_0} \rho_f}$$

No free charges $\equiv$ No bound charges in linear dielectrics

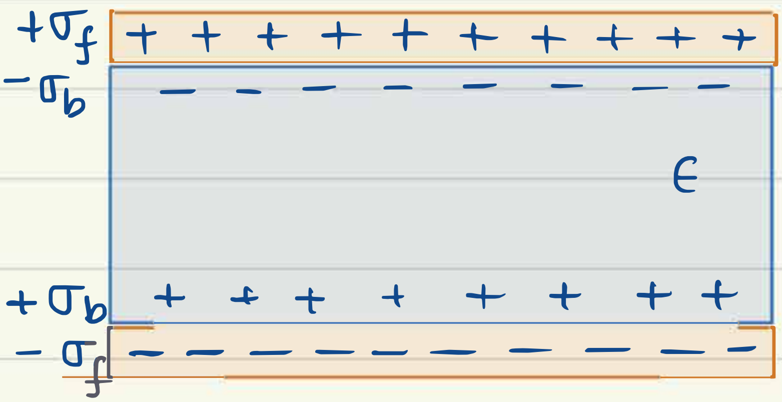
Capacitor with a linear Dielectric.



bare capacitor

$$E = \frac{V}{d} = \frac{\sigma_f}{\epsilon_0}$$

$$\Rightarrow V = \frac{\sigma_f}{\epsilon_0} d \qquad C = \frac{\epsilon_0 A}{d}$$



$\pm \sigma_f$ is constant but dielectric filling space

$$E' = \frac{V'}{d} = \frac{\sigma_f - \sigma_b}{\epsilon_0} \Rightarrow E' < E$$

$$\bar{P} = \epsilon_0 \chi_0 \bar{E}' \qquad \frac{\epsilon}{\epsilon_0} = 1 + \chi_0$$

$$\Rightarrow \sigma_b = \bar{P} \cdot \hat{n} = \epsilon_0 \chi_0 E' \qquad \Rightarrow E' = \frac{\sigma_b}{\epsilon_0 \chi_0}$$

$$\Rightarrow \sigma_f - \sigma_b = \sigma_f / (1 + \chi_0) \quad \text{and} \quad E' = \frac{\sigma_f}{\epsilon_0 (1 + \chi_0)} = \frac{\sigma_f}{\epsilon}$$

$$\Rightarrow V' = \frac{\sigma_f d}{\epsilon} \qquad \because \epsilon > \epsilon_0$$

$$\Rightarrow V' < V$$

$$\Rightarrow C' = \frac{\epsilon A}{d} = \frac{\epsilon_0 \epsilon_r A}{d}$$

For constant charge the voltage is smaller

$\Rightarrow$ At constant voltage the dielectric capacitor can store more charge $\Rightarrow$ higher Capacitance

$$C' > C$$

Energy stored in the Capacitor:

Bare: $W = \frac{\epsilon_0}{2} \int E^2 d\tau$

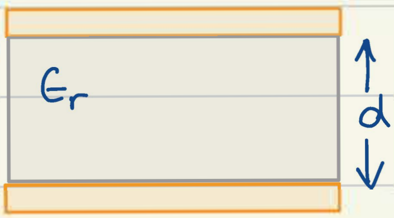
Dielectric: $W' = \frac{1}{2} \int \bar{E} \cdot \bar{D} \, d\tau = \frac{\epsilon}{2} \int E^2 d\tau$

$\Rightarrow$ Energy stored increases by a factor ϵ_r

High-k dielectrics   Aim: Increase capacitance

$C = \frac{\epsilon_0 A}{d}$ (vacuum) $\longrightarrow$ $\frac{\epsilon A}{d}$ (dielectrics)

SiO_2 is the industry standard for gate dielectric



$$C = \frac{\epsilon A}{d} = \frac{\epsilon_0 \epsilon_r A}{d}$$

$\epsilon_r(SiO_2) \sim 3.6$ $d \sim 1\,nm$

$\epsilon_r(\text{high k}) \sim 50$ $d \sim 5\,nm$

small separation ≡ high leakage current (primarily tunnelling)

$$\frac{C(\text{high k})}{C(SiO_2)} \sim \frac{50}{5} \times \frac{1}{3.6} = 2.78$$

Low-k Dielectric   Aim: Increase insulation

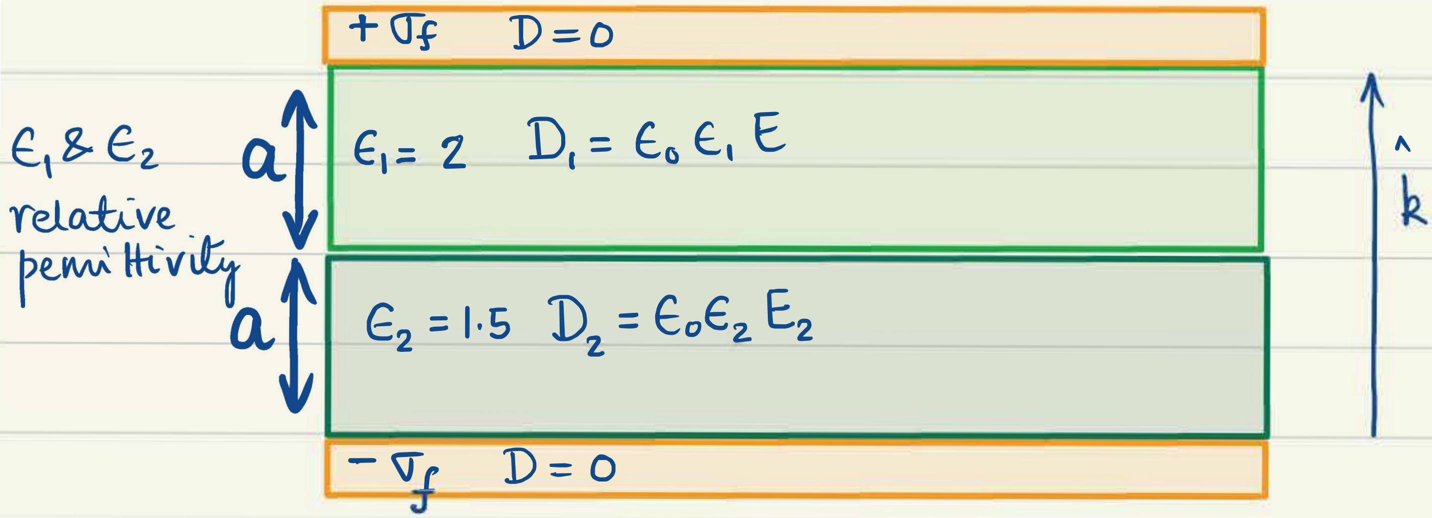
Best insulation is vacuum $\epsilon_r = 1$ $(\epsilon = \epsilon_0)$

Vacuum does not provide mechanical support

$\Rightarrow$ need of material $\epsilon_r < \epsilon_r(SiO_2)$

typical separation of insulation $\sim 200\,nm$

Ex: Space between the plates of a // plate capacitor is filled as shown. Find $\bar{D}, \bar{E}, \bar{P}, \rho_b, \sigma_b$.



(i) $\bar{D} = ?$ $\oint \bar{D} \cdot \bar{da} = Q_f$

Gaussian Box at the metal-dielectric interface

$$-D_1 A = \sigma_f A \hat{k} \Rightarrow \bar{D}_1 = -\sigma_f \hat{k} \quad \text{(top plate)}$$

$$D_2 A = -\sigma_f A \hat{k} \Rightarrow \bar{D}_2 = -\sigma_f \hat{k} \quad \text{(bottom plate)}$$

$$\Rightarrow E_1 = D_1 / \epsilon_1 \epsilon_0 \quad \text{and} \quad \bar{E}_2 = D_2 / \epsilon_2 \epsilon_0$$

$$\Rightarrow \bar{E}_1 = -\frac{\sigma_f}{2\epsilon_0} \hat{k} \qquad \bar{E}_2 = -\frac{2\sigma_f}{3\epsilon_0} \hat{k}$$

$$\Rightarrow \bar{P}_1 = \epsilon_0 \chi_1 \bar{E}_1 = \epsilon_0 (\epsilon_1 - 1) \left(-\frac{\sigma_f}{2\epsilon_0}\right) \hat{k} = -\frac{\sigma_f}{2} \hat{k}$$

$$\bar{P}_2 = \epsilon_0 (\epsilon_2 - 1) \left(-\frac{2\sigma_f}{3\epsilon_0}\right) k = -\frac{\sigma_f}{3} \hat{k}$$

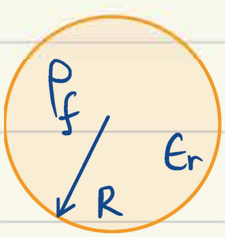
Potential difference between plates

$$V = -\int_{\text{bottom}}^{\text{top}} \bar{E}\cdot dl = E_1 a + E_2 a - \frac{7\sigma_f a}{6\epsilon_0}$$

$\because$ $\bar{\nabla}\cdot\bar{P}_1 = \bar{\nabla}\cdot\bar{P}_2 = 0$ $\qquad \rho_b = 0$ everywhere

$$\sigma_b = \bar{P}\cdot\hat{n} \Rightarrow \quad \sigma_{b1} = \pm\sigma/2$$
$$\sigma_{b2} = \pm\sigma/3$$

Ex: A sphere of linear dielectric (ϵ_r) has embedded in it a uniform charge density ρ_f.
Calculate the potential at centre of sphere.



$$V(r=0) = -\int_{\infty}^{r} \bar{E}\cdot\bar{dl} \quad = \frac{\rho R^2}{3\epsilon_0}\left(1+\frac{1}{2\epsilon_r}\right)$$

$$\oint \bar{D}\cdot\bar{da} = Q_f$$

Take 2 gaussian surfaces
① $r<R$ and ② $r>R$

Calculate $\bar{D}_{inside}$ and $\bar{D}_{outside}$ $\rightarrow$ $\bar{E}_{inside} = \frac{\bar{D}_{inside}}{\epsilon_0\epsilon_r}$

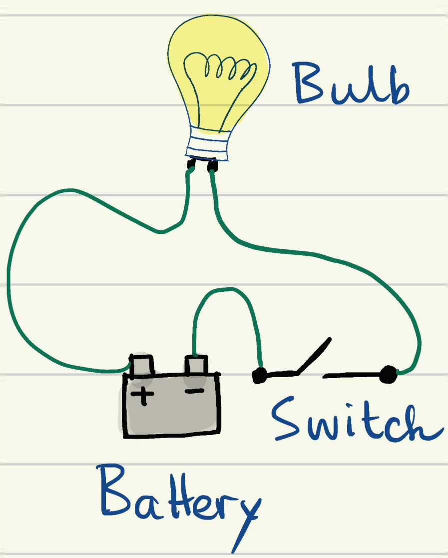
$$\bar{E}_{outside} = \frac{\bar{D}_{outside}}{\epsilon_0}$$

And Finally....

Electrostatic Forces and Interactions rule a great part of the world.

From the forces at a gecko's feet, that helps it climb a vertical wall, the forces that bind the electrons to an atomic nuclei → to the stability of the most extravagant man made structures electrostatic forces are at play.

Consider a circuit with light bulb.



At $t=0$ the Switch is closed. How long does it take for the bulb to turn on?